

NONPARAMETRIC CONDITIONAL LOCAL INDEPENDENCE TESTING

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Conditional local independence is an asymmetric independence relation among continuous time stochastic processes. It describes whether the evolution of one process is directly influenced by another process given the histories of additional processes, and it is important for the description and learning of causal relations among processes. We develop a model-free framework for testing the hypothesis that a counting process is conditionally locally independent of another process. To this end, we introduce a new functional parameter called the Local Covariance Measure (LCM), which quantifies deviations from the hypothesis. Following the principles of double machine learning, we propose an estimator of the LCM and a test of the hypothesis using nonparametric estimators and sample splitting or cross-fitting. We call this test the (cross-fitted) Local Covariance Test ((X)-LCT), and we show that its level and power can be controlled uniformly, provided that the nonparametric estimators are consistent with modest rates. We illustrate the theory by an example based on a marginalized Cox model with time-dependent covariates, and we show in simulations that when double machine learning is used in combination with cross-fitting, then the test works well without restrictive parametric assumptions.

1. Introduction. Notions of how one variable influences a target variable are central to both predictive and causal modeling. Depending on the objective, the relevant notion of influence can be variable importance in a predictive model of the target, but it can also be the causal effect of the variable on the target. In either case, we can investigate influence conditionally on a third variable—to quantify the added predictive value, the direct causal effect or the causal effect adjusted for a confounder. Our interests are in an asymmetric notion of direct influence among stochastic processes, which is not adequately captured by classical (symmetric) notions of conditional dependence. The objective of this paper is therefore to quantify this notion of asymmetric influence and specifically to develop a new nonparametric test of the hypothesis that one stochastic process does not directly influence another.

The hypothesis we consider is that of *local independence*—a concept introduced by Schweder (1970) for Markov processes as a continuous time formalization of the phenomenon that the past of one stochastic process does not directly influence the evolution of another stochastic process. Generalizations to other continuous time processes were given by Aalen (1987) and studied by Commenges and Gégout-Petit (2009), who systematically used the term *conditional local independence* for the general concept. We will in this paper follow that convention whenever we want to emphasize the conditional nature of the local independence. We note that (conditional) local independence is a continuous time version of the discrete time concept of Granger noncausality (Aalen (1987), Granger (1969)).

To illustrate the concept of conditional local independence, we will in this Introduction consider an example involving three processes: X , Z and N ; see Figure 1. The process N is

Received April 2022; revised August 2023.

MSC2020 subject classifications. Primary 62G10; secondary 62G20.

Key words and phrases. Nonparametric inference, local independence, double machine learning, functional CLT, stochastic processes.

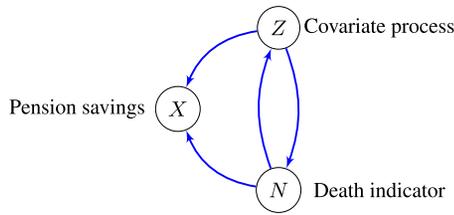


FIG. 1. *Local independence graph illustrating a dependence structure among the three processes X , Z and N . Here, N is the indicator of death for an individual, X is their cumulative pension savings and Z is a covariate process. All nodes in this graph have implicit self-loops. There is no edge from X to N , which indicates that death is not directly influenced by pension savings. This can be formalized as N being conditionally locally independent of X , which is the hypothesis we aim to test.*

the indicator of death, $N_t = \mathbb{1}(T \leq t)$, for an individual with survival time T , and X_t denotes the total pension savings of the individual at time t . The process Z is a covariate process, for example, health variables or employment status, that may directly affect both the pension savings and the survival time. This is indicated in Figure 1 by edges pointing from Z to X and N . Edges pointing from N to X and Z indicate that a death event directly affects both X and Z (which take the values X_T and Z_T , respectively, after time T , see Section 2.2).

To define conditional local independence, let $\mathcal{F}_t^{N,Z} = \sigma(N_s, Z_s; s \leq t)$ denote the filtration generated by the N - and Z -processes. The σ -algebra $\mathcal{F}_t^{N,Z}$ represents the information contained in the N - and the Z -processes before time t . Informally, the process N_t is conditionally locally independent of the process X_t given $\mathcal{F}_t^{N,Z}$ if $(X_s)_{s \leq t}$ does not add predictable information to $\mathcal{F}_t^{N,Z}$ about the infinitesimal evolution of N_t . For this particular example, this means that the conditional hazard function of T does not depend on $(X_s)_{s \leq t}$ given $\mathcal{F}_t^{N,Z}$. In Figure 1, the hypothesis of interest, that N_t is conditionally locally independent of X_t given $\mathcal{F}_t^{N,Z}$, is represented by the lack of an edge from X to N .

A systematic investigation of algebraic properties of conditional local independence was initiated by Didelez (2007, 2008, 2015). She also introduced local independence graphs, such as the directed graph in Figure 1, to graphically represent all conditional local independencies among several processes, and she studied the semantics of these graphs. This work was extended further by Mogensen and Hansen (2020) to graphical representations of partially observed systems. While we will not formally discuss local independence graphs, the problem of learning such graphs from data was an important motivation for us to develop a non-parametric test of conditional local independence. A constraint-based learning algorithm of local independence graphs was given by Mogensen, Malinsky and Hansen (2018) in terms of a conditional local independence oracle, but a practical algorithm requires that the oracle is replaced by conditional local independence tests.

Another important motivation for considering conditional local independence arises from causal models. With a structural assumption about the stochastic process specification, a conditional local independence has a causal interpretation (Aalen (1987), Aalen et al. (2012), Commenges and Gégout-Petit (2009)), and if the causal stochastic system is completely observed, a test of conditional local independence is a test of no *direct* causal effect. See also Røysland et al. (2022), who use local independence graphs to formulate criteria for identification of causal effects in continuous-time survival models. If the causal stochastic system is only partially observed, a conditional local dependency need not correspond to a direct causal effect due to unobserved confounding, but the projected local independence graph, as introduced by Mogensen and Hansen (2020), retains a causal interpretation, and its Markov equivalence class can be learned by conditional local independence testing. In addition, within the

framework of structural nested models, testing the hypothesis of no *total* causal effect can also be cast as a test of conditional local independence (Lok (2008), Theorem 9.2).

To appreciate what conditional local independence means—and, in particular, what it does not mean—it is useful to compare with ordinary conditional independence. In our example, N_t is conditionally locally independent of X_t given $\mathcal{F}_t^{N,Z}$, but this implies neither that $N \perp\!\!\!\perp X|Z$ (as processes), nor that $N_t \perp\!\!\!\perp X_t|\mathcal{F}_t^Z$. In fact, these conditional independencies cannot hold in this example where $X_t = X_T$ for $t \geq T$ —except in special cases such as T being a deterministic function of Z . Theorem 2 in Didelez (2008) gives a sufficient condition for $N_t \perp\!\!\!\perp X_t|\mathcal{F}_t^Z$ to hold in terms of the local independence graph, but this condition is also not fulfilled by the graph in Figure 1 due to the edge from N to X . Moreover, conditional local independence is in general also different from the baseline conditional independence $T \perp\!\!\!\perp X_0|Z_0$ unless both X and Z are time-independent; see Section 3.2. In Section E in the Supplementary Material (Christgau, Petersen and Hansen (2023)), we elaborate further upon the connection to semiparametric survival models. Didelez (2008) argues that N_t being conditionally locally independent of X_t given $\mathcal{F}_t^{N,Z}$ heuristically means that $N_t \perp\!\!\!\perp \mathcal{F}_t^X|\mathcal{F}_t^{N,Z}$, but this is technically problematic in continuous time. If T has a continuous distribution, then for any fixed t , $N_t = N_{t-}$ almost surely, whence N_t is almost surely $\mathcal{F}_{t-}^{N,Z}$ -measurable and conditionally independent of anything given $\mathcal{F}_{t-}^{N,Z}$. This heuristic can thus not be used to formally define conditional local independence in continuous time. See instead the formal Definition 2 by Didelez (2008) or our Definition 2.1.

Several examples from health sciences given by Didelez (2008) demonstrate the usefulness of conditional local independence for multivariate event systems, and more recent attention to event systems in the machine learning community (Achab et al. (2017), Bacry et al. (2017), Cai et al. (2022), Xu, Farajtabar and Zha (2016), Zhou, Zha and Song (2013)) testifies to the relevance of conditional local independence. This line of research relies primarily on the linear Hawkes process model, which is effectively used to infer local independence graphs—sometimes even interpreted causally. The Hawkes model is attractive because conditional local independencies can be inferred from corresponding kernel functions being zero—and statistical tests can readily be based on parametric or nonparametric estimation of kernels.

A less attractive property of the Hawkes model is that it is not closed under marginalization. As with any model based statistical test, the validity of the test is jeopardized by model misspecification, hence even within a subsystem of a linear Hawkes process, a test of conditional local independence based on a Hawkes model may be invalid.

The challenges resulting from model misspecification and marginalization is investigated further in Sections 2.2 and 6 based on an extension of our introductory example and Cox’s survival model. Both the Hawkes model and the Cox model illustrate that conditional local independence might be expressed and tested within a (semi)parametric model, but model misspecification makes us question the validity of any such model based test. Thus, there is a need for a nonparametric test of the hypothesis of conditional local independence. Moreover, since we cannot translate the hypothesis into an equivalent hypothesis about classical conditional independence, we cannot directly use existing nonparametric tests, such as the GCM (Shah and Peters (2020)) or the GHCM (Lundborg, Shah and Peters (2022)), of conditional independence.

We propose a new nonparametric test when the target process N is a counting process and X is a real valued process, and where the hypothesis is that N is conditionally locally independent of X given a filtration \mathcal{F}_t . In the context of the introductory example, $\mathcal{F}_t = \mathcal{F}_t^{N,Z}$. We consider a counting process target primarily because the theory of conditional local independence is most complete in this case, but generalizations are possible—we refer to the discussion in Section 7. Within our framework, we base our test on an infinite-dimensional parameter, which we call the Local Covariance Measure (LCM). It is a function of time,

which is constantly equal to zero under the hypothesis. Our main result is that the LCM can be estimated by using the ideas of double machine learning (Chernozhukov et al. (2018)) in such a way that the estimator converges uniformly at a \sqrt{n} -rate to a mean zero Gaussian martingale under the hypothesis of conditional local independence. We use the LCM to develop the (cross-fitted) Local Covariance Test ((X)-LCT), for which we derive uniform level and power results.

1.1. *Organization of the paper.* In Section 2, we introduce the general framework for formulating the hypothesis of conditional local independence. This includes the introduction in Section 2.1 of an abstract residual process, which is used to define the LCM as a functional target parameter indexed by time. The LCM equals the zero-function under the hypothesis of conditional local independence, and to test this hypothesis we introduce an estimator of the LCM in Section 2.3. The estimator is a stochastic process, and we describe how sample splitting is to be used for its computation via the estimation of two unknown components.

In Section 3, we give interpretations of the LCM and its estimator. We show that the LCM estimator is a Neyman orthogonalized score statistic in Section 3.1, and in Section 3.2 we relate LCM to the partial copula when X is time-independent.

In Section 4, we state the main results of the paper. We establish in Section 4.1 that the LCM estimator generally approximates the LCM with an error of order $n^{-1/2}$. Under the hypothesis of conditional local independence, we show that the (scaled) LCM estimator converges weakly to a mean zero Gaussian martingale. The estimator requires a model of the target process N as well as the process X conditionally on \mathcal{F}_t to achieve the orthogonalization at the core of double machine learning. The model of X is in this paper expressed indirectly in terms of the residual process, and we show that if we can learn the residual process at rate $g(n)$ and the model of N at rate $h(n)$ such that $g(n), h(n) \rightarrow 0$ and $\sqrt{n}g(n)h(n) \rightarrow 0$ for $n \rightarrow \infty$ then we achieve a \sqrt{n} -rate convergence of the LCM estimator. We also show that the variance function of the Gaussian martingale can be estimated consistently, and we give a general result on the asymptotic distribution of univariate test statistics based on the LCM estimator. All asymptotic results are presented in the framework of uniform stochastic convergence.

Section 5 gives explicit examples of univariate test statistics, including the local covariance test based on the normalized supremum of the LCM estimator. Its asymptotic distribution is derived and we present results on uniform asymptotic level and power. In Section 5.2, we present the generalization from the sample split estimator to the cross-fit estimator. Though this estimator and the corresponding cross-fit Local Covariance Test (X-LCT) are a bit more involved to compute and analyze, X-LCT is more powerful, and thus our recommended test for practical usage.

The survival example from the Introduction is used and elaborated upon throughout the paper. We introduce a Cox model in terms of the time-varying covariate processes, and we report in Section 6 the results from a simulation study based on this model.

The paper is concluded by a discussion in Section 7.

The Supplementary Material (Christgau, Petersen and Hansen (2023)), henceforth referred to as the supplement, consists of Sections A through G and contains: proofs of results in this paper (A); definitions and results on uniform asymptotics (B); a uniform version of Rebolledo's martingale CLT (C); an overview of achievable rate results for estimation of nuisance parameters that enter into the LCM estimator (D); a comparison with semiparametric survival models (E); details on Neyman orthogonality (F) and additional results from the simulation study (G).

2. The local covariance measure. In this section, we present the general framework of the paper, we define conditional local independence and we introduce the local covariance measure as a means to quantify deviations from conditional local independence. In Section 2.3, we outline how the local covariance measure can be estimated using double machine learning and sample splitting. We illustrate the central concepts and methods by an example based on Cox’s survival model with time-varying covariates.

We consider a counting process $N = (N_t)$ and another real value process $X = (X_t)$, both defined on the probability space $(\Omega, \mathbb{F}, \mathbb{P})$. All processes are assumed to be defined on a common compact time interval. We assume, without loss of generality, that the time interval is $[0, 1]$. We will assume that N is adapted w.r.t. a right continuous and complete filtration \mathcal{F}_t , and we denote by \mathcal{G}_t the right continuous and complete filtration generated by \mathcal{F}_t and X_t . We assume throughout that X is càglàd (i.e., has sample paths that are continuous from the left and with limits from the right), which will ensure bounded sample paths and that the process is \mathcal{G}_t -predictable.

In the survival example of the Introduction, $N_t = \mathbb{1}(T \leq t)$ is the indicator of whether death has happened by time t , and there can only be one event per individual observed. Furthermore, $\mathcal{F}_t = \mathcal{F}_t^{N,Z}$ and $\mathcal{G}_t = \mathcal{F}_t^{N,X,Z}$. Our general setup works for any counting process, thus it allows for recurrent events and censoring, and the filtration \mathcal{F}_t can contain the histories of any number of processes in addition to the history of N itself.

2.1. *The hypothesis of conditional local independence.* The counting process N is assumed to have an \mathcal{F}_t -intensity λ_t , that is, λ_t is \mathcal{F}_t -predictable and with

$$\Lambda_t = \int_0^t \lambda_s ds$$

being the compensator of N ,

$$(1) \quad M_t = N_t - \Lambda_t$$

is a local \mathcal{F}_t -martingale. Within this framework, we can define the hypothesis of conditional local independence precisely.

DEFINITION 2.1 (Conditional local independence). We say that N_t is conditionally locally independent of X_t given \mathcal{F}_t if the local \mathcal{F}_t -martingale M_t defined by (1) is also a local \mathcal{G}_t -martingale.

For simplicity, we may also refer to this hypothesis as *local independence* and write

$$(2) \quad H_0 : M_t = N_t - \Lambda_t \quad \text{is a local } \mathcal{G}_t\text{-martingale.}$$

As argued in the Introduction, the hypothesis of local independence is the hypothesis that observing X on $[0, t]$ does not add any information to \mathcal{F}_{t-} about whether an N -event will happen in an infinitesimal time interval $[t, t + dt)$. Definition 2.1 captures this interpretation by requiring that the \mathcal{F}_t -compensator, Λ , of N is also the \mathcal{G}_t -compensator. Thus, λ is also the \mathcal{G}_t -intensity under H_0 .

If N has \mathcal{G}_t -intensity λ , the innovation theorem, Theorem II.T14 in Brémaud (1981), gives that the predictable projection $\lambda_t = \mathbb{E}(\lambda_t | \mathcal{F}_{t-})$ is the (predictable) \mathcal{F}_t -intensity. Local independence follows if λ is \mathcal{F}_t -predictable. Intensities are, however, only unique almost surely, and we can have local independence even if λ is not *a priori* \mathcal{F}_t -predictable but have an \mathcal{F}_t -predictable version. When N has \mathcal{G}_t -intensity λ , H_0 is thus equivalent to λ having an \mathcal{F}_t -predictable version. We find Definition 2.1 preferable because it directly gives an operational criterion for determining whether N has an \mathcal{F}_t -predictable version of a \mathcal{G}_t -intensity.

REMARK 2.2 (Censoring). Suppose that the data is censored such that $(N_t, X_t, \mathcal{F}_t) = (N_{t \wedge C}^*, X_{t \wedge C}^*, \mathcal{F}_{t \wedge C}^*)$, where $(N^*, X^*, \mathcal{F}^*)$ are uncensored data and where C is the censoring time. The hypothesis regarding the uncensored data,

$$H_0^*: N_t^* \text{ is locally independent of } X_t^* \text{ given } \mathcal{F}_t^*,$$

might then be the hypothesis of interest. If $\mathbb{1}(C \geq t)$ happens to be \mathcal{F}_t^* -predictable, it is straightforward to show that H_0^* implies H_0 , and consequently a test of H_0 is also a test of H_0^* . However, \mathcal{F}_t^* may not *a priori* contain information about the censoring process. Suppose instead that the common condition of *independent censoring* (Andersen et al. (1993)) holds, which is equivalent to N_t^* being locally independent of $\mathcal{C}_t := \mathbb{1}(C \leq t)$ given \mathcal{G}_t^* (Røysland et al. (2022)). Then H_0^* implies that N_t is locally independent of X_t given $\mathcal{F}_t \vee \mathcal{F}_t^C$. Thus, in order to test H_0^* , we replace \mathcal{F}_t by the enlarged filtration $\mathcal{F}_t \vee \mathcal{F}_t^C$ and proceed *mutatis mutandis* with testing H_0 using the observed data.

Since X is assumed càglàd, and thus especially \mathcal{G}_t -predictable, the stochastic integral

$$(3) \quad \int_0^t X_s \, dM_s,$$

is under H_0 a local \mathcal{G}_t -martingale. A test could be based on detecting whether (3) is, indeed, a local martingale. We will take a slightly different approach where we replace the integrand X by a residual process as defined below. We do so for two reasons. First, to achieve a \sqrt{n} -rate via double machine learning we need the integrand to fulfill (4) below. Second, other choices of integrands than X could potentially lead to more powerful tests.

DEFINITION 2.3 (Residual process). A residual process $G = (G_t)_{t \in [0,1]}$ of X_t given \mathcal{F}_t is a càglàd stochastic process that is \mathcal{G}_t -adapted and satisfies

$$(4) \quad \mathbb{E}(G_t | \mathcal{F}_{t-}) = 0, \quad t \in [0, 1].$$

The geometric interpretation is that the residual process evolves such that G_t is orthogonal to $L_2(\mathcal{F}_{t-})$ within $L_2(\mathcal{G}_{t-})$ at each time t . One obvious residual process is the *additive residual process* given by

$$G_t = X_t - \Pi_t = X_t - \mathbb{E}(X_t | \mathcal{F}_{t-}),$$

where $\Pi_t = E(X_t | \mathcal{F}_{t-})$ denotes the predictable projection of the càglàd process X_t ; see Theorem VI.19.2 in (Rogers and Williams (2000)). The additive residual projects X_t onto the orthogonal complement of $L_2(\mathcal{F}_{t-})$, but this may not necessarily remove all \mathcal{F}_t -predictable information from X_t . An alternative choice that does so under sufficient regularity conditions is the *quantile residual process* given by

$$G_t = F_t(X_t) - \frac{1}{2},$$

where F_t is the conditional distribution function given by $F_t(x) = \mathbb{P}(X_t \leq x | \mathcal{F}_{t-})$. The quantile residual process satisfies (4) provided that $(t, x) \mapsto F_t(x)$ is continuous. In Section 3.1, we discuss additional transformations of X that can also be applied before any residualization procedure.

We will formulate the general results in terms of an abstract residual process, but we focus on the additive residual process in the examples. Any nondegenerate residual process will contain a predictive model of (aspects of) X_t given \mathcal{F}_{t-} in order to satisfy (4). We use \hat{G}_t to denote the residual obtained by plugging in an estimate of that predictive model. For the additive residual process, the predictive model is Π_t and $\hat{G}_t = X_t - \hat{\Pi}_t$. For the quantile residual process, the predictive model is F_t and $\hat{G}_t = \hat{F}_t(X_t) - \frac{1}{2}$.

We can now define our functional target parameter of interest, which we call the *local covariance measure*.

DEFINITION 2.4 (Local covariance measure). With G_t a residual process, define for $t \in [0, 1]$,

$$(5) \quad \gamma_t = \mathbb{E}(I_t), \quad \text{where } I_t = \int_0^t G_s \, dM_s,$$

whenever the expectation is well-defined. We call the function $t \mapsto \gamma_t$ the Local Covariance Measure (LCM).

The following propositions illuminate how γ relates to the null hypothesis of N_t being conditionally locally independent of X_t given \mathcal{F}_t .

PROPOSITION 2.5. Under H_0 , the process $I = (I_t)$ is a local \mathcal{G}_t -martingale with $I_0 = 0$. If I is a martingale, then $\gamma_t = 0$ for $t \in [0, 1]$.

To interpret γ in the alternative, we assume that N has \mathcal{G}_t -intensity λ .

PROPOSITION 2.6. If $\int_0^1 \mathbb{E}(|G_s|(\lambda_s + \lambda_s)) \, ds < \infty$, then for every $t \in [0, 1]$,

$$\gamma_t = \int_0^t \text{cov}(G_s, \lambda_s - \lambda_s) \, ds.$$

In particular, γ is the zero-function if and only if $\text{cov}(G_s, \lambda_s - \lambda_s) = 0$ for almost all $s \in [0, 1]$.

We note that under H_0 , the condition $\int_0^1 \mathbb{E}(|G_s| \lambda_s) \, ds < \infty$ is sufficient to ensure that I is a martingale and $\gamma_t = 0$ for all $t \in [0, 1]$. By Proposition 2.6, the LCM quantifies deviations from H_0 in terms of the covariance between the residual process and the difference of the \mathcal{F}_t - and \mathcal{G}_t -intensities. To this end, note that if X happens to be \mathcal{F}_t -adapted, then $\mathcal{G}_t = \mathcal{F}_t$ and N is trivially locally independent of X . The hypothesis of local independence is only of interest when \mathcal{G}_t is a strictly larger filtration than \mathcal{F}_t , that is, when X provides information not already in \mathcal{F}_t .

For the additive residual process, where $G_t = X_t - \Pi_t$,

$$\gamma_t = \mathbb{E}\left(\int_0^t G_s \, dM_s\right) = \mathbb{E}\left(\int_0^t X_s \, dM_s\right) - \mathbb{E}\left(\int_0^t \Pi_s \, dM_s\right)$$

provided that the expectations are well-defined. Since the predictable projection Π_t has a càglàd version and is \mathcal{F}_t -predictable, and since M_t is a local \mathcal{F}_t -martingale, $\int_0^t \Pi_s \, dM_s$ is a local \mathcal{F}_t -martingale. If it is a martingale, it is a mean zero martingale, and

$$(6) \quad \gamma_t = \mathbb{E}\left(\int_0^t X_s \, dM_s\right) = \mathbb{E}\left(\sum_{\tau \leq t: \Delta N_\tau = 1} X_\tau - \int_0^t X_s \lambda_s \, ds\right).$$

The computation above shows that the additive residual process defines the same functional target parameter γ_t as the stochastic integral (3) would. It is, however, the representation of γ_t as the expectation of the residualized stochastic integral that will allow us to achieve a \sqrt{n} -rate of convergence of the estimator of γ_t in cases where the estimator of λ_t converges at a slower rate.

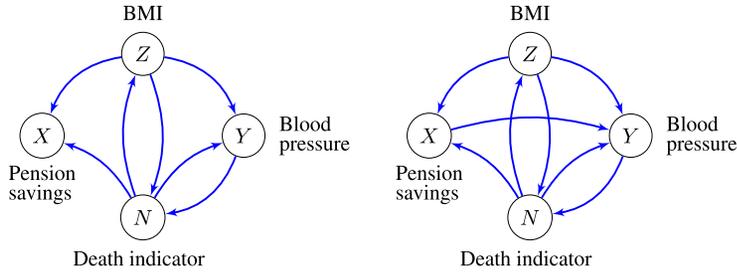


FIG. 2. Local independence graphs illustrating how the three processes X, Y and Z could affect each other and time of death in the Cox example. There is no direct influence of X (pension savings) on time of death in either of the two graphs, but in the left graph the death indicator is furthermore conditionally locally independent of X given the history of Z and N . In the right graph, Z and N does not block all paths from X to N , thus conditioning on the history of Z and N only would not render N conditionally locally independent of X .

2.2. A Cox model with a partially observed covariate process. To further illustrate the hypothesis of conditional local independence and the local covariance measure we consider an example based on Cox’s survival model with time dependent covariates. This is an extension of the example from the Introduction with T being the time to death of an individual, and with X and Z being time-varying processes. There is, moreover, one additional time-varying process Y in the full model.

An interpretation of the processes is as follows:

- X = Pension savings,
- Y = Blood pressure,
- Z = BMI.

Periods of overweight or obesity may influence blood pressure in the long term, and due to, for example, job market discrimination, high BMI could influence pension savings negatively. Death risk is influenced directly by BMI and blood pressure but not the size of your pension savings. Figure 2 illustrates two possible dependence structures among the three processes and the death time as local independence graphs, and we will use these two graphs to discuss the concept of conditional local independence of pension savings on time to death.

We assume that $T \in [0, 1]$ and that X, Y and Z have continuous sample paths. Recall also that $N_t = \mathbb{1}(T \leq t)$ is the death indicator process. To maintain some form of realism, all processes are stopped at time of death, that is, $X_t = X_T, Y_t = Y_T$ and $Z_t = Z_T$ for $t \geq T$. This feedback from the death event to the other processes is reflected in Figure 2 by the edges pointing out of N . Recall also that

$$\mathcal{F}_t^{N,Z} = \sigma(N_s, Z_s; s \leq t)$$

is the filtration generated by the N - and Z -processes. We use a similar notation for other processes and combinations of processes. For example, $\mathcal{F}_t^{N,X,Y,Z}$ is the filtration generated by N and all three X -, Y - and Z -processes. With λ_t^{full} denoting the $\mathcal{F}_t^{N,X,Y,Z}$ -intensity of time of death based on the history of all processes, we assume in this example a Cox model given by

$$(7) \quad \lambda_t^{\text{full}} = \mathbb{1}(T \geq t) \lambda_t^0 e^{Y_t + \beta Z_t}$$

with λ_t^0 a deterministic baseline intensity. It is not important that λ_t^{full} is a Cox model for our general theory, but it allows for certain theoretical computations in this example.

The fact that λ_t^{full} does not depend upon X_t implies that λ_t^{full} is also the $\mathcal{F}_t^{N,Y,Z}$ -intensity, and according to Definition 2.1, N_t is conditionally locally independent of X_t given $\mathcal{F}_t^{N,Y,Z}$.

This is in agreement with the local independence graphs in Figure 2 where there is no edge in either of them from X to N .

We will take an interest in the case where Y is unobserved and test the hypothesis:

$$H_0: N_t \text{ is conditionally locally independent of } X_t \text{ given } \mathcal{F}_t^{N,Z}.$$

That is, with Y unobserved we want test if the intensity of time to death given the history of N , X and Z depends on X . To simplify notation, let $\mathcal{F}_t = \mathcal{F}_t^{N,Z}$ and $\mathcal{G}_t = \mathcal{F}_t^{N,X,Z}$, in accordance with the general notation. The \mathcal{G}_t -intensity is by the innovation theorem given as

$$(8) \quad \lambda_t = \mathbb{E}(\lambda_t^{\text{full}} | \mathcal{G}_{t-}) = \mathbb{1}(T \geq t) \lambda_t^0 e^{\beta Z_t} \mathbb{E}(e^{Y_t} | \mathcal{G}_{t-}),$$

while the \mathcal{F}_t -intensity is

$$(9) \quad \lambda_t = \mathbb{E}(\lambda_t^{\text{full}} | \mathcal{F}_{t-}) = \mathbb{1}(T \geq t) \lambda_t^0 e^{\beta Z_t} \mathbb{E}(e^{Y_t} | \mathcal{F}_{t-}),$$

and H_0 is equivalent to $\lambda_t = \lambda_t$ almost surely. Comparing (8) and (9), we see that H_0 holds in this example if $\mathbb{E}(e^{Y_t} | \mathcal{G}_{t-}) = \mathbb{E}(e^{Y_t} | \mathcal{F}_{t-})$, and a sufficient condition for this is

$$(10) \quad \mathcal{F}_t^X \perp\!\!\!\perp \mathcal{F}_t^Y | \mathcal{F}_t.$$

The condition (10) is in concordance with the left graph in Figure 2 (see Theorem 2 in Didelez (2008)), but not the right, and it implies H_0 . We will in Section 6.1 elaborate on condition (10) and give explicit examples.

We recall that H_0 can be reformulated as λ_t not depending on X , and we could investigate the hypothesis via a marginal Cox model

$$(11) \quad \lambda_t^{\text{cox}} = \mathbb{1}(T \geq t) \lambda_t^0 e^{\alpha_1 X_t + \alpha_2 Z_t}$$

and test if $\alpha_1 = 0$. The Cox model is, however, not closed under marginalization and the semiparametric model (11) is quite likely misspecified. Consequently, a test of $\alpha_1 = 0$ is not equivalent to a test of H_0 .

Our proposed nonparametric test of H_0 does not rely on a specific (semi)parametric model of λ_t . To test H_0 , we consider the LCM using the additive residual process. Then (6) implies that

$$\gamma_t = \mathbb{E}\left(X_T N_t - \int_0^t X_s \lambda_s \, ds\right).$$

By Proposition 2.5, $\gamma_t = 0$ for $t \in [0, 1]$ under H_0 , whence conditional local independence implies $\gamma_t = 0$, and we test H_0 by estimating γ_t and testing if it is constantly equal to 0.

Before introducing a general estimator of the LCM in Section 2.3, we outline how to estimate the end point parameter γ_1 in this example. Due to $T \leq 1$ and the appearance of the indicator $\mathbb{1}(T \geq t)$ in (9),

$$\gamma_1 = \mathbb{E}\left(X_T - \int_0^T X_s \lambda_s \, ds\right).$$

With i.i.d. observations $(T_1, X_1, Z_1), \dots, (T_n, X_n, Z_n)$ and (nonparametric) estimates, $\hat{\lambda}_{j,t}$, based on $(T_1, Z_1), \dots, (T_n, Z_n)$, we could compute the plug-in estimate

$$\hat{\gamma}_{1,\text{plug-in}}^{(n)} = \frac{1}{n} \sum_{j=1}^n \left(X_{j,T_j} - \int_0^{T_j} X_{j,s} \hat{\lambda}_{j,s} \, ds \right).$$

However, we cannot expect the plug-in estimator to have a \sqrt{n} -rate unless $\hat{\lambda}$ has \sqrt{n} -rate, which effectively requires parametric model assumptions on the intensity. Using the definition of γ_1 in terms of the additive residual process $G_t = X_t - \Pi_t$, we also have that

$$(12) \quad \gamma_1 = \mathbb{E}\left(X_T - \Pi_T - \int_0^T (X_s - \Pi_s) \lambda_s \, ds\right).$$

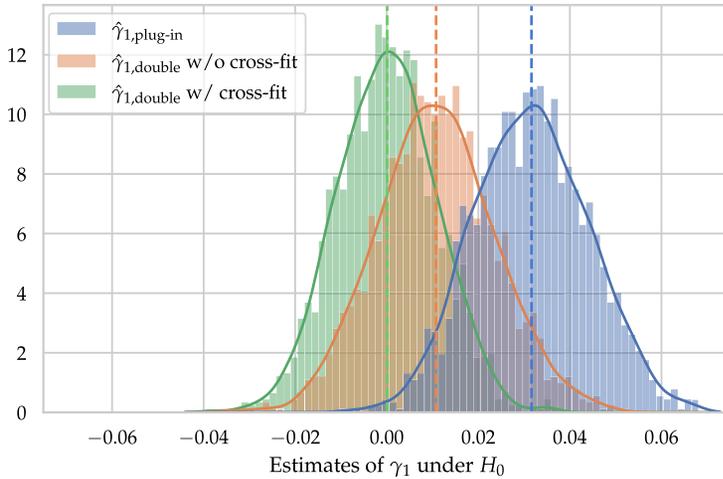


FIG. 3. Histograms of the distributions of three different estimators of γ_1 . Each histogram contains 1000 estimates fitted to samples of size $n = 500$. The samples were sampled from a model that satisfies the hypothesis of conditional local independence and hence the ground truth is $\gamma_1 = 0$. See Section 6.2 for further details of the data generating process.

A double machine learning estimator based on the ideas by Chernozhukov et al. (2018) is therefore obtained by plugging in two nonparametric estimators:

$$\hat{\gamma}_{1,\text{double}}^{(n)} = \frac{1}{n} \sum_{j=1}^n \left(X_{j,T_j} - \hat{\Pi}_{j,T_j} - \int_0^{T_j} (X_{j,s} - \hat{\Pi}_{j,s}) \hat{\lambda}_{j,s} ds \right).$$

To achieve a small bias and a \sqrt{n} -rate of convergence, we use sample splitting. The nonparametric estimates $\hat{\Pi}_j$ and $\hat{\lambda}_j$ are based on one part of the sample only, and are thus independent of the other part of the sample used for testing; see Section 2.3. To obtain a fully efficient estimator, multiple sample splits can be combined, for example, via cross-fitting; see Section 5.2.

Figure 3 shows the distributions of $\hat{\gamma}_{1,\text{plug-in}}^{(500)}$ and $\hat{\gamma}_{1,\text{double}}^{(500)}$ for the Cox example with $\gamma_1 = 0$; see Section 6.2 for details on the full model specification. The latter estimator was computed using cross-fitting but also without using any form of sample splitting. The figure illustrates the bias of $\hat{\gamma}_{1,\text{plug-in}}^{(500)}$, which is somewhat diminished by double machine learning without sample splitting and mostly eliminated by double machine learning in combination with cross-fitting.

2.3. Estimating the local covariance measure. To estimate the LCM, we assume that we have observed n i.i.d. replications of the processes, $(N_1, X_1, \mathcal{F}_1), \dots, (N_n, X_n, \mathcal{F}_n)$, where observing $\mathcal{F}_j = (\mathcal{F}_{j,t})$ signifies that anything adapted to the j th filtration is computable from observations. The process N_j is adapted to \mathcal{F}_j , while X_j is not, and \mathcal{G}_j denotes the smallest right continuous and complete filtration generated by X_j and \mathcal{F}_j .

For each n , we consider a sample split corresponding to a partition $J_n \cup J_n^c = \{1, \dots, n\}$ of the indices into two disjoint sets. We let $\hat{\lambda}^{(n)}$ and $\hat{G}^{(n)}$ be estimates of the intensity and the residualization map, respectively, fitted on data indexed by J_n^c only. By an estimate, $\hat{\lambda}^{(n)}$, of λ we mean a (stochastic) function that can be evaluated on the basis of $\mathcal{F}_{j,t}$ for $j \in J_n$, and its value, denoted by $\hat{\lambda}_{j,t}^{(n)}$, is interpreted as a prediction of $\lambda_{j,t}$. The stochasticity in $\hat{\lambda}^{(n)}$ arises from its dependence on data indexed by J_n^c , from which its functional form is completely determined. Similarly, $\hat{G}^{(n)}$ is a function that can be evaluated on the basis of $\mathcal{G}_{j,t}$ for $j \in J_n$

Algorithm 1: Sample split estimator of LCM

- 1 **input:** processes $(N_j, X_j, Z_j)_{j=1, \dots, n}$, partition $J_n \cup J_n^c$ of indices ;
- 2 **options:** historical regression methods for estimation of λ and G given N and Z ,
- 3 discrete time grid $0 = t_0 < \dots < t_k \leq 1$;
- 4 **begin**
- 5 historically regress $(X_j)_{j \in J_n^c}$ on $(N_j, Z_j)_{j \in J_n^c}$ to obtain a fitted model $\hat{G}^{(n)}$;
- 6 historically regress $(N_j)_{j \in J_n^c}$ on $(N_j, Z_j)_{j \in J_n^c}$ to obtain a fitted model $\hat{\lambda}^{(n)}$;
- 7 compute out of sample residuals $\hat{G}_{j,t_i}^{(n)}$ and $\hat{M}_{j,t_i}^{(n)}$ for $j \in J_n$ and $i = 0, \dots, k$;
- 8 for each $i = 1, \dots, k$, compute

$$\tilde{\gamma}_i^{(n)} = \frac{1}{|J_n|} \sum_{j \in J_n} \sum_{1 \leq l \leq i} \hat{G}_{j,t_l}^{(n)} (\hat{M}_{j,t_l}^{(n)} - \hat{M}_{j,t_{l-1}}^{(n)})$$

9 **end**

10 **output:** Local covariance measure $\tilde{\gamma}^{(n)}$ numerically approximated on grid;

to give a prediction $\hat{G}_{j,t}^{(n)}$ of $G_{j,t}$. In Section 6.1, we illustrate through the Cox example how $\hat{\lambda}^{(n)}$ and $\hat{G}^{(n)}$ are to be computed in practice when we use sample splitting. In Section D in the supplement, we give more examples of such estimation procedures and discuss their statistical properties in greater detail.

To ease notation, we will throughout assume that (N, X, \mathcal{F}) denotes one additional process and filtration—independent of and with the same distribution as the observed processes. Then the estimated intensity $\hat{\lambda}^{(n)}$ and estimated residual process $\hat{G}^{(n)}$ can be evaluated on (N, X, \mathcal{F}) , and thus we may write $\hat{\lambda}_t^{(n)}$ and $\hat{G}_t^{(n)}$ to denote template copies of $\hat{\lambda}_{j,t}^{(n)}$ and $\hat{G}_{j,t}^{(n)}$ for $j \in J_n$.

In terms of the estimates $\hat{\lambda}^{(n)}$ and $\hat{G}^{(n)}$, we estimate LCM by the stochastic process $\hat{\gamma}^{(n)}$ given by

$$(13) \quad \hat{\gamma}_t^{(n)} = \frac{1}{|J_n|} \sum_{j \in J_n} \int_0^t \hat{G}_{j,s}^{(n)} d\hat{M}_{j,s}^{(n)},$$

where $\hat{M}_{j,t}^{(n)} = N_{j,t} - \int_0^t \hat{\lambda}_{j,s}^{(n)} ds$. We can regard $\hat{\gamma}_t^{(n)}$ as a double machine learning estimator of γ_t , with the observations indexed by J_n^c used to learn models of λ and G , and with observations indexed by J_n used to estimate γ_t based on these models. In Section 5.2, we define the more efficient estimator that uses cross-fitting, but it is instructive to study the simpler estimator based on sample splitting first.

In practical applications, we do not directly observe the filtration \mathcal{F}_j , but rather samples from the stochastic processes generating the filtration. In accordance with the introductory Cox example, consider \mathcal{F}_j and \mathcal{G}_j given by $\mathcal{F}_{j,t} = \sigma(Z_{j,s}, N_{j,s}; s \leq t)$ and $\mathcal{G}_{j,t} = \sigma(X_{j,s}, Z_{j,s}, N_{j,s}; s \leq t)$ for a third stochastic process Z_j , with Z_j possibly being multivariate. Within this setup, a general procedure for numerically computing the LCM is described in Algorithm 1. Here, historical regression refers to any method, which regresses the outcome at a given time on the history of the regressors up to that time. For example, historical linear regression is discussed in Section 6 and various alternative methods are discussed in Section D in the supplement. The choice of sample split will be discussed further in Section 5.2 in the context of cross-fitting.

As in Section 2.2, we could suggest estimating the entire function $t \mapsto \gamma_t$ by a simple plug-in estimator of λ using the representation (6). Figure 4 illustrates the distribution of

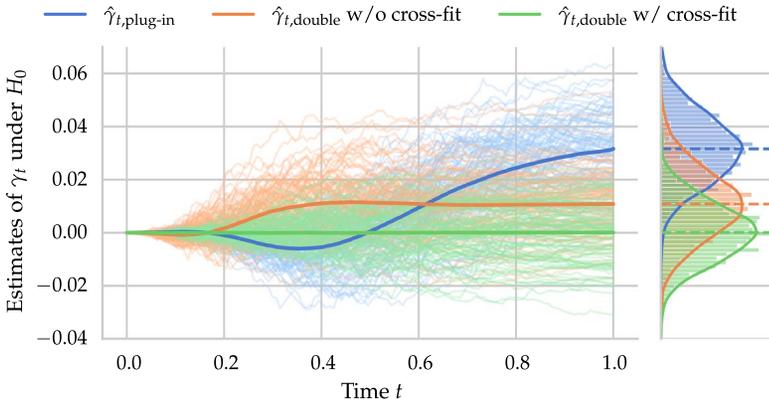


FIG. 4. A time dependent extension of Figure 3 showing the distribution of the sample paths $t \mapsto \hat{\gamma}_{t,\text{plug-in}}^{(500)}$ and $t \mapsto \hat{\gamma}_{t,\text{double}}^{(500)}$, the latter with and without using cross-fitting. The data were simulated under H_0 where $t \mapsto \gamma_t$ is the zero function. See Section 6.2 for further details of the data generating process.

estimators of the entire time dependent LCM for this plug-in estimator together with the double machine learning estimator with and without using cross-fitting. The figure also shows the distribution of the endpoint being the same distribution shown in Figure 3. The simulation is under H_0 , and we see that only the double machine learning estimator with cross-fitting results in estimated sample paths centered around 0.

3. Interpretations of the LCM estimator. In this section, we provide some additional perspectives on and interpretations of the LCM. First, we show that the LCM estimator can be seen as a Neyman orthogonalization of the score statistic for a particular one-parameter family. The abstract formulation of the residual process (G_t) permits that we transform X into another \mathcal{G}_t -predictable processes. Using this perspective, we may optimize the choice of the process X in terms of power.

Next, we show that when X is independent of time, the test statistic reduces in a survival context to certain covariance measures between X -residuals and Cox–Snell residuals, which we can link to existing test statistics for ordinary conditional independence.

3.1. *Neyman orthogonalization of a score statistic.* Consider the one-parameter family of \mathcal{G}_t -intensities

$$\lambda_t^\beta = e^{\beta X_t} \lambda_t$$

for $\beta \in \mathbb{R}$. Within this one-parameter family, the hypothesis of conditional local independence is equivalent to $H_0 : \beta = 0$. The normalized log-likelihood with n i.i.d. observations in the interval $[0, t]$ is

$$\begin{aligned} \ell_t(\beta) &= \frac{1}{n} \sum_{j=1}^n \left(\int_0^t \log(\lambda_{j,s}^\beta) dN_{j,s} - \int_0^t \lambda_{j,s}^\beta ds \right) \\ &= \frac{1}{n} \sum_{j=1}^n \left(\int_0^t \beta X_{j,s} + \log(\lambda_{j,s}) dN_{j,s} - \int_0^t e^{\beta X_{j,s}} \lambda_{j,s} ds \right). \end{aligned}$$

Straightforward computations show that

$$\partial_\beta \ell_t(0) = \frac{1}{n} \sum_{j=1}^n \int_0^t X_{j,s} dM_{j,s} \quad \text{and} \quad -\partial_\beta^2 \ell_t(0) = \frac{1}{n} \sum_{j=1}^n \int_0^t X_{j,s}^2 \lambda_{j,s} ds.$$

If λ were known, the score statistic $\partial_\beta \ell_t(0)$ satisfies $\mathbb{E}(\partial_\beta \ell_t(0)) = \gamma_t$. Moreover, under $H_0 : \beta = 0$ we have that $-\partial_\beta^2 \ell_t(0) = \langle \partial_\beta \ell_t(0) \rangle$ is a consistent estimate of the asymptotic variance of the mean zero martingale $\partial_\beta \ell_t(0)$. The hypothesis of local independence—with λ known—could thus be tested using the score test statistic $-\partial_\beta \ell_t(0)^2 / \partial_\beta^2 \ell_t(0)$.

The nuisance parameter λ is, however, unknown and we want to avoid restrictive parametric assumptions about λ . Replacing $X_{j,t}$ by the residual process $G_{j,t}$ in the score statistic $\partial_\beta \ell_t(0)$ gives a *Neyman orthogonalized* score

$$\frac{1}{n} \sum_{j=1}^n \int_0^t G_{j,s} dM_{j,s}.$$

This score is linear in λ , which is used in supplementary Section F to show that it satisfies the Neyman orthogonality condition under H_0 ; cf. Definition 2.1 in Chernozhukov et al. (2018). In this section, it is also shown that the act of replacing X_t with $G_t = X_t - \Pi_t$ can, in fact, be viewed as *concentrating out* the intensity of the score statistic in the sense of Newey (1994). While Neyman orthogonality is never invoked explicitly, it is implicitly a central part of the asymptotic results for the LCM estimator (in particular Lemma A.7 in the supplement).

The perspective on the LCM from a Neyman orthogonalized score statistic suggests that a test based on the LCM has most power against alternatives in the one-parameter family λ^β . If it happens that the most important alternatives are of the form

$$\lambda_t^\beta = e^{\beta \bar{X}_t} \lambda_t$$

for some \mathcal{G}_t -predictable process \bar{X}_t different from X_t , then we should replace X_t by \bar{X}_t in our test statistic, that is, in the residualization procedure. Examples of processes \bar{X}_t are

- transformations, $\bar{X}_t = f(X_t)$ for a function f
- time-shifts, $\bar{X}_t = X_{t-s}$ for $s > 0$
- linear filters, $\bar{X}_t = \int_0^t \kappa(t-s) X_s ds$ for a kernel κ
- nonlinear filters, $\bar{X}_t = \phi(\int_0^t \kappa(t-s) f(X_s) ds)$ for a kernel κ and functions f and ϕ .

Any finite number of such processes could, of course, also be combined into a vector process, and we could, indeed, generalize the LCM estimator (13) to a vector process. The generalization is straightforward.

3.2. Survival time with time-independent covariates. A different perspective on the test statistic is obtained if X is constant over time, if $N_t = \mathbb{1}(T \leq t)$ is the counting process of a survival time T , and if $\mathcal{F}_t = \sigma(N_s, Z; s \leq t)$ where Z is a vector of additional baseline variables. Then the \mathcal{F}_t -intensity is

$$\lambda_t = \mathbb{1}(T \geq t) h(t, Z),$$

where $h(t, Z)$ is the hazard function for T given the baseline Z . In this special case, the hypothesis of conditional local independence is equivalent to the ordinary conditional independence

$$(14) \quad X \perp\!\!\!\perp T | Z.$$

We also find that

$$\gamma_t = \mathbb{E}(X(\mathbb{1}(T \leq t) - \Lambda_{t \wedge T})),$$

and in particular $\gamma_1 = \mathbb{E}(X(1 - \Lambda_T))$ as $T \in [0, 1]$ by assumption. Since Λ_T is exponentially distributed with mean 1, we may write

$$\gamma_1 = -\text{cov}(X, \Lambda_T).$$

Testing if $\gamma_1 \neq 0$ in this particular setup is effectively a test of the conditional independence (14). When (14) is true, it further holds that $\Pi_t = \mathbb{E}(X|\mathcal{F}_{t-}) = \mathbb{E}(X|Z) = \Pi_0$ is independent of t , and if we incorporate this into our model of Π , the LCM estimator of γ_1 equals

$$(15) \quad \hat{\gamma}_1^{(n)} = \frac{1}{|J_n|} \sum_{j \in J_n} (X_j - \hat{\Pi}_{j,0})(1 - \hat{\Lambda}_{T_j}).$$

This is a (nonnormalized) generalized covariance measure (GCM), see (Shah and Peters (2020)), which is simply the (negative) empirical covariance between the additive residuals $X_j - \hat{\Pi}_{j,0}$ and the Cox–Snell residuals $\hat{\Lambda}_{T_j}$.

Alternatively, consider the quantile residual process $G_{j,t} = F_t(X_j) - \frac{1}{2}$ where $F_t(x) = \mathbb{P}(X \leq x|\mathcal{F}_{t-})$. If (14) is true, it holds again that $F_t(x) = F_0(x) = \mathbb{P}(X \leq x|Z)$ is independent of t , and our LCM estimator becomes

$$(16) \quad \hat{\gamma}_1^{(n)} = \frac{1}{|J_n|} \sum_{j \in J_n} \hat{G}_{j,0}(1 - \hat{\Lambda}_{T_j}).$$

This is likewise an empirical covariance, but now between the generalized residuals and the Cox–Snell residuals. This is closely related to the partial copula between X and T given Z , which can be estimated as

$$\frac{1}{|J_n|} \sum_{j \in J_n} \hat{G}_{j,0} \left(\frac{1}{2} - \exp(-\hat{\Lambda}_{T_j}) \right).$$

See Petersen and Hansen (2021) for further details on the partial copula and how this statistic can be used to test conditional independence.

Under a combined rate condition on estimation of G and Λ , the endpoint statistics above are known to be asymptotically Gaussian with mean zero when the hypothesis of conditional independence in (14) holds. Within this survival setting, the endpoint statistics (15) can furthermore be seen as a score test derivable from a semiparametric efficient score function. Section E in the supplement gives the details for two specific semiparametric survival models.

Whenever $\hat{G}_{j,t} = \hat{G}_{j,0}$ is independent of time, for example, if we incorporate (14) into the residual model, the t -indexed LCM estimator is

$$\hat{\gamma}_t^{(n)} = \frac{1}{|J_n|} \sum_{j \in J_n} \hat{G}_{j,0} (\mathbb{1}(T_j \leq t) - \hat{\Lambda}_{T_j \wedge t}),$$

which can be seen as a t -indexed extension of (16). For a general, time-dependent residual process, the full t -indexed LCM estimator is

$$\hat{\gamma}_t^{(n)} = \frac{1}{|J_n|} \sum_{j \in J_n} \mathbb{1}(T_j \leq t) \hat{G}_{j,T_j} - \int_0^{t \wedge T_j} \hat{G}_{j,s} \hat{\lambda}_s ds.$$

The general results of this paper show that the t -indexed LCM estimator is asymptotically distributed as a mean zero Gaussian martingale under H_0 . This appears to be a novel result even when X is constant over time. However, the main contributions of this paper is to the case where X and Z are stochastic processes varying with time—where the hypothesis of conditional local independence is also distinct from (14).

4. General asymptotic results. In this section, we derive uniform asymptotic results regarding the general LCM estimator as a stochastic process. In Section 5, we discuss how to construct tests of H_0 based on the asymptotic results.

We assume that N has a \mathcal{G}_t -intensity λ_t , we let $\Lambda_t = \int_0^t \lambda_s ds$ denote the \mathcal{G}_t -compensator of N and let $\mathbf{M}_t = N_t - \Lambda_t$ be the compensated local \mathcal{G}_t -martingale. We also recall that $\hat{\gamma}^{(n)}$ denotes the LCM estimator based on sample splitting as defined in Section 2.3. Within this framework, we consider the decomposition

$$(17) \quad \sqrt{|J_n|} \hat{\gamma}^{(n)} = U^{(n)} + R_1^{(n)} + R_2^{(n)} + R_3^{(n)} + D_1^{(n)} + D_2^{(n)},$$

where the processes $U^{(n)}, R_1^{(n)}, R_2^{(n)}, R_3^{(n)}, D_1^{(n)}$ and $D_2^{(n)}$ are given by

$$(18) \quad U_t^{(n)} = \frac{1}{\sqrt{|J_n|}} \sum_{j \in J_n} \int_0^t G_{j,s} d\mathbf{M}_{j,s},$$

$$(19) \quad R_{1,t}^{(n)} = \frac{1}{\sqrt{|J_n|}} \sum_{j \in J_n} \int_0^t G_{j,s} (\lambda_{j,s} - \hat{\lambda}_{j,s}^{(n)}) ds,$$

$$(20) \quad R_{2,t}^{(n)} = \frac{1}{\sqrt{|J_n|}} \sum_{j \in J_n} \int_0^t (\hat{G}_{j,s}^{(n)} - G_{j,s}) d\mathbf{M}_{j,s},$$

$$(21) \quad R_{3,t}^{(n)} = \frac{1}{\sqrt{|J_n|}} \sum_{j \in J_n} \int_0^t (\hat{G}_{j,s}^{(n)} - G_{j,s}) (\lambda_{j,s} - \hat{\lambda}_{j,s}^{(n)}) ds,$$

$$(22) \quad D_{1,t}^{(n)} = \frac{1}{\sqrt{|J_n|}} \sum_{j \in J_n} \int_0^t G_{j,s} (\lambda_{j,s} - \lambda_{j,s}) ds,$$

$$(23) \quad D_{2,t}^{(n)} = \frac{1}{\sqrt{|J_n|}} \sum_{j \in J_n} \int_0^t (\hat{G}_{j,s}^{(n)} - G_{j,s}) (\lambda_{j,s} - \lambda_{j,s}) ds.$$

Note that the processes D_1 and D_2 are (almost surely) the zero-process under H_0 , since the null is equivalent to λ_t being a version of λ_t . We proceed to show that

- the processes $U^{(n)}$ and $D_1^{(n)} - \sqrt{|J_n|} \gamma$ each converge in distribution,
- and the processes R_1, R_2, R_3, D_2 converge to the zero process.

For the analysis of each of R_1, R_2 and D_2 , sample splitting is used to render the summands conditionally independent.

These asymptotic properties imply that $\sqrt{|J_n|}(\hat{\gamma}^{(n)} - \gamma)$ is stochastically bounded in general, so the LCM estimator will asymptotically detect if the LCM is nonzero. Moreover, it will follow that $U^{(n)}$ drives the asymptotic limit of the LCM estimator under H_0 . Based on these general asymptotic results, we derive in Section 5 asymptotic error control for tests based on the LCM estimator.

4.1. Asymptotics of the LCM estimator. Our asymptotic results are formulated in terms of uniform stochastic convergence, which has also been discussed extensively in the recent literature on hypothesis testing (Lundborg et al. (2022), Lundborg, Shah and Peters (2022), Neykov, Balakrishnan and Wasserman (2021), Scheidegger, Hörrmann and Bühlmann (2022), Shah and Peters (2020)). Uniform convergence allows us to establish uniform asymptotic level of our proposed test, as well as power under local alternatives. We have collected key definitions and results related to uniform convergence in Section B in the supplement.

To state uniform assumptions and asymptotic results, we need to indicate a range of possible sampling distributions for which the assumptions apply and the results hold. For this purpose, we extend our setup and allow all data to be parametrized by a fixed parameter set Θ . The set Θ is not *a priori* assumed to have any structure, and $\theta \in \Theta$ simply indicates that $N^\theta, X^\theta, \lambda^\theta, G^\theta$, etc. have θ -dependent distributions. We generally denote evaluation of processes or derived quantities for a specific θ -value by a superscript, with the LCM, γ^θ , in particular, depending on θ . The LCM estimator is likewise written as $\hat{\gamma}^{(n),\theta} = (\hat{\gamma}_t^{(n),\theta})$ for $\theta \in \Theta$ to denote its dependence on the sampling distribution. The superscript notation is, however, heavy and unnecessary in many cases and we will suppress the dependency on $\theta \in \Theta$ whenever it is not needed. Any result that does not explicitly involve Θ should be understood as a pointwise result for each $\theta \in \Theta$.

The parametrization allows us to express convergence in distribution and probability uniformly over Θ , which are denoted by $\xrightarrow{D/\Theta}$ and $\xrightarrow{P/\Theta}$, respectively. These concepts are defined rigorously in Definition B.2 in the supplement. We note that uniform convergence reduces to classical (pointwise) convergence if Θ is a singleton, which corresponds to fixing the sampling distribution. We also introduce the parameter subset

$$(24) \quad \Theta_0 := \{\theta \in \Theta \mid H_0 \text{ is valid}\},$$

consisting of all parameter values for which the hypothesis of conditional local independence holds. Correspondingly, we will use $\xrightarrow{D/\Theta_0}$ and $\xrightarrow{P/\Theta_0}$ to denote stochastic convergences uniformly over Θ_0 .

We are now ready to formulate the underlying assumptions on the data required for our asymptotic results. These assumptions may appear strong, but we argue in the discussion in Section 7 that they are not unreasonable from a practical viewpoint.

ASSUMPTION 4.1. There exist constants $C, C' > 0$, such that for any parameter value $\theta \in \Theta$:

- (i) The \mathcal{G}_t^θ -intensity λ_t^θ of N^θ is càglàd with $\sup_{0 \leq t \leq 1} \lambda_t^\theta \leq C$ almost surely.
- (ii) The residual process G^θ is càglàd with $\sup_{0 \leq t \leq 1} |G_t^\theta| \leq C'$ almost surely.

The estimator, $\hat{\lambda}_t^{(n)}$, of λ_t and the estimator, $\hat{G}_t^{(n)}$, of the residual process are assumed to satisfy the same bounds as λ_t and G_t . We note that Assumption 4.1(i) implies that \mathbf{M}_t is a true \mathcal{G}_t -martingale, and by the innovation theorem, $\lambda_t = \mathbb{E}(\lambda_t \mid \mathcal{F}_{t-})$. As a consequence, the \mathcal{F}_t -intensity λ_t inherits the boundedness from the \mathcal{G}_t -intensity λ_t , and M_t is an \mathcal{F}_t -martingale. More generally, we have the following proposition ensuring that stochastic integrals are true martingales, for example, that I_t is a martingale under H_0 .

PROPOSITION 4.1. Under Assumption 4.1, it holds that each of the processes

$$\left(\int_0^t f(G_s) d\mathbf{M}_s \right)_{t \in [0,1]} \quad \text{and} \quad \left(\int_0^t f(\hat{G}_s^{(n)}) d\mathbf{M}_s \right)_{t \in [0,1]}$$

are mean zero, square integrable \mathcal{G}_t -martingales for any $f \in C(\mathbb{R})$.

To express the asymptotic distribution of $U^{(n)}$, we need its variance function.

DEFINITION 4.2. We define the variance function $\mathcal{V}: [0, 1] \rightarrow [0, \infty]$ as

$$(25) \quad \mathcal{V}(t) = \mathbb{E} \left(\int_0^t G_s^2 dN_s \right).$$

As everything else, the variance function, $\mathcal{V} = \mathcal{V}^\theta$, is also indexed by the parameter θ , which we, for notational simplicity, suppress unless explicitly needed.

By taking $f(x) = x^2$ in Proposition 4.1, Assumption 4.1 implies that for each $t \in [0, 1]$,

$$\mathcal{V}(t) = \mathbb{E} \left(\int_0^t G_s^2 \lambda_s \, ds \right) < \infty.$$

Moreover, $\mathcal{V}(t)$ is the variance of $\int_0^t G_s \, d\mathbf{M}_s$, which under H_0 is the same as the variance of $I_t = \int_0^t G_s \, dM_s$.

With the assumptions above, we can prove the following proposition about the uniform distributional limit of the process $U^{(n)}$ in the Skorokhod space $D[0, 1]$, the space of càdlàg functions from $[0, 1]$ to \mathbb{R} endowed with the Skorokhod topology. A corresponding pointwise result is an application of Rebolledo’s classical martingale CLT. Our generalization to uniform convergence is based on a uniform extension of Rebolledo’s theorem; see Theorem C.4 in Section C in the supplement.

PROPOSITION 4.3. *Under Assumption 4.1, it holds that*

$$U^{(n),\theta} \xrightarrow{D/\Theta} U^\theta$$

in $D[0, 1]$ as $n \rightarrow \infty$, where for each $\theta \in \Theta$, U^θ is a mean zero continuous Gaussian martingale on $[0, 1]$ with variance function \mathcal{V}^θ .

To control the remainder terms in (17), we will bound the estimation errors in terms of the 2-norm, $\|\cdot\|_2$, on $L_2([0, 1] \times \Omega)$, that is,

$$\|W\|_2^2 = \mathbb{E} \left(\int_0^1 W_s^2 \, ds \right)$$

for any process $W \in L_2([0, 1] \times \Omega)$. We will make the following consistency assumptions on $\hat{\lambda}^{(n)}$ and $\hat{G}^{(n)}$.

ASSUMPTION 4.2. Assume that $|J_n| \rightarrow \infty$ when $n \rightarrow \infty$ and let

$$g^\theta(n) = \left\| G^\theta - \hat{G}^{(n),\theta} \right\|_2 \quad \text{and} \quad h^\theta(n) = \left\| \lambda^\theta - \hat{\lambda}^{(n),\theta} \right\|_2.$$

Then each of the sequences $g^\theta(n)$, $h^\theta(n)$, and $\sqrt{|J_n|} g^\theta(n) h^\theta(n)$ converge to zero uniformly over Θ as $n \rightarrow \infty$, that is,

$$\lim_{n \rightarrow \infty} \sup_{\theta \in \Theta} \max \{ g^\theta(n), h^\theta(n), \sqrt{|J_n|} g^\theta(n) h^\theta(n) \} = 0.$$

With this assumption, we can establish that the remainder terms also converge uniformly to the zero process.

PROPOSITION 4.4. *Under Assumptions 4.1 and 4.2, it holds that*

$$\sup_{t \in [0,1]} |R_{i,t}^{(n),\theta}| \xrightarrow{P/\Theta} 0$$

as $n \rightarrow \infty$ for $i = 1, 2, 3$.

To control the asymptotic behavior of the LCM estimator in the alternative, we need to control the two terms $D_1^{(n)}$ and $D_2^{(n)}$.

PROPOSITION 4.5. *Let Assumptions 4.1 and 4.2 hold true.*

(i) *The stochastic process $D_1^{(n),\theta} - \sqrt{|J_n|}\gamma^\theta$ converges in distribution in $(C[0, 1], \|\cdot\|_\infty)$ uniformly over Θ as $n \rightarrow \infty$.*

(ii) *If $G_t^\theta = X_t^\theta - \Pi_t^\theta$ is the additive residual process, then $D_2^{(n),\theta} \xrightarrow{P/\Theta} 0$ in $D[0, 1]$ as $n \rightarrow \infty$.*

We note that $D_2^{(n)}$ might not vanish without an assumption like G_t being the additive residual process, and it is not clear if $D_2^{(n)}$ will even converge in general. We will not pursue an analysis of the asymptotic behavior of $D_2^{(n)}$ in the general case. We note, however, that if we can estimate G with a parametric rate, that is, $\sqrt{|J_n|}g(n) = O(1)$. Then it follows from the Cauchy–Schwarz inequality that $D_2^{(n)}$ is stochastically bounded, and $D_1^{(n)}$ still dominates in the alternative where $\gamma \neq 0$.

We can combine all of the propositions into a single theorem regarding the asymptotics of the LCM estimator, which we consider as our main result.

THEOREM 4.6. *Let Assumptions 4.1 and 4.2 hold true.*

(i) *It holds that*

$$\sqrt{|J_n|}\hat{\gamma}^{(n),\theta} \xrightarrow{D/\Theta_0} U^\theta$$

in $D[0, 1]$ as $n \rightarrow \infty$, where for each $\theta \in \Theta_0$, U^θ is a mean zero continuous Gaussian martingale on $[0, 1]$ with variance function \mathcal{V}^θ .

(ii) *For the additive residual process, it holds that for every $\varepsilon > 0$ there exists $K > 0$ such that*

$$(26) \quad \limsup_{n \rightarrow \infty} \sup_{\theta \in \Theta} \mathbb{P}(\sqrt{|J_n|} \cdot \|\hat{\gamma}^{(n),\theta} - \gamma^\theta\|_\infty > K) < \varepsilon.$$

Thus, we have established the weak asymptotic limit of $\sqrt{|J_n|}\hat{\gamma}^{(n)}$ under H_0 . However, the variance function \mathcal{V} of the limiting Gaussian martingale is unknown and must be estimated from data. We propose to use the empirical version of (25),

$$(27) \quad \hat{\mathcal{V}}_n(t) = \frac{1}{|J_n|} \sum_{j \in J_n} \int_0^t (\hat{G}_{j,s}^{(n)})^2 dN_{j,s} = \frac{1}{|J_n|} \sum_{j \in J_n} \sum_{\tau \leq t: \Delta N_{j,s}=1} (\hat{G}_{j,\tau}^{(n)})^2,$$

for which we have the following consistency result.

PROPOSITION 4.7. *Under Assumptions 4.1 and 4.2, it holds that*

$$\sup_{t \in [0, 1]} |\hat{\mathcal{V}}_n^\theta(t) - \mathcal{V}^\theta(t)| \xrightarrow{P/\Theta} 0,$$

as $n \rightarrow \infty$.

We emphasize that \mathcal{V} is only the asymptotic variance function of the LCM estimator under H_0 . It is always the asymptotic variance function of $U^{(n)}$, but in the alternative the asymptotic distribution of $\hat{\gamma}^{(n)}$ also involves the asymptotic distribution of $D_1^{(n)}$ and is thus more complicated.

Tests of conditional local independence can now be constructed in terms of univariate functionals of $\hat{\gamma}^{(n)}$ and $\hat{\mathcal{V}}_n$ that quantify the magnitude of the LCM. The asymptotics of such test statistics under H_0 are described in the following corollary, which is essentially an application of the continuous mapping theorem.

COROLLARY 4.8. *Let $\mathcal{J}: D[0, 1] \times D[0, 1] \rightarrow \mathbb{R}$ be a functional that is continuous on the closed subset $C[0, 1] \times \{\mathcal{V}^\theta: \theta \in \Theta_0\}$ with respect the uniform topology, that is, the topology generated by the norm $\|(f_1, f_2)\| = \max\{\|f_1\|_\infty, \|f_2\|_\infty\}$ for $f_1, f_2 \in D[0, 1]$. Define the test statistic*

$$\hat{D}_n^\theta = \mathcal{J}(\sqrt{|J_n|}\hat{\gamma}^{(n),\theta}, \hat{\mathcal{V}}_n^\theta).$$

Under Assumptions 4.1 and 4.2, it holds that

$$(28) \quad \hat{D}_n^\theta \xrightarrow{\mathcal{D}/\Theta_0} \mathcal{J}(U^\theta, \mathcal{V}^\theta), \quad n \rightarrow \infty,$$

where U^θ is a mean zero continuous Gaussian martingale with variance function \mathcal{V}^θ .

5. The local covariance test. In this section, we introduce a practically applicable test based on the LCM estimator. Using the asymptotic distribution of the LCM estimator, we show that the asymptotic distribution of our proposed test is independent of the sampling distribution under H_0 and has an explicit representation. We show, in addition, uniform asymptotic level of the test, and we give a uniform power result for the additive residual process. Finally, we modify the test to be based on a cross-fitted estimator of the LCM instead of using sample splitting, and we show uniform level of that test.

To construct a test statistic based on the LCM estimator, it is beneficial that its distributional limit does not depend on the variance function. As a simple example, consider the endpoint test statistic:

$$(29) \quad (\hat{\mathcal{V}}_n(1))^{-\frac{1}{2}} \sqrt{|J_n|} \hat{\gamma}_1^{(n)},$$

which under H_0 converges in distribution to $\mathcal{V}(1)^{-\frac{1}{2}} U_1$ by Corollary 4.8. The distribution of the latter is the standard normal distribution, and in particular it does not depend on \mathcal{V} .

Any test statistic constructed from $\hat{\gamma}^{(n)}$ should capture deviations of γ_t away from 0. The test statistic in (29) does, however, only consider the endpoint of the process, and since γ is not necessarily monotone, γ_t may deviate more from 0 for other $t \in [0, 1]$. Thus, in order to increase power against such alternatives we consider the test statistic

$$(30) \quad \hat{T}_n = \frac{\sqrt{|J_n|}}{\sqrt{\hat{\mathcal{V}}_n(1)}} \sup_{0 \leq t \leq 1} |\hat{\gamma}_t^{(n)}|.$$

We refer to \hat{T}_n as the *Local Covariance Test statistic* (LCT statistic). We proceed to show that the LCT statistic can be turned into a test of H_0 with asymptotic level α , and which has asymptotic power against any alternative with a nonzero LCM. This is the best we can hope for of any test based on the LCM estimator.

We note that it might be possible to establish similar results for other norms of the LCM, for example, a statistic based on a weighted L_2 -norm. However, since other norms of the distributional limit U will generally have a distribution with a complicated dependency on \mathcal{V} , we believe that the LCT statistic is the simplest to construct.

To establish uniform asymptotic level via Corollary 4.8 for tests based on test statistics such as (30), we need to assume that the asymptotic variances in $t = 1$ are uniformly bounded away from zero.

ASSUMPTION 5.1. There exists a $\delta_1 > 0$ such that for all $\theta \in \Theta$ it holds that $\mathcal{V}^\theta(1) \geq \delta_1$.

5.1. *Type I and type II error control.* We proceed to show that under H_0 , the LCT statistic is distributed as $\sup_{0 \leq t \leq 1} |B_t|$, where (B_t) is a standard Brownian motion. From this point onward, we let S denote a random variable with such a distribution and note that its CDF can be written as

$$(31) \quad F_S(x) = \mathbb{P}(S \leq x) = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \exp\left(-\frac{\pi^2(2k+1)^2}{8x^2}\right), \quad x > 0.$$

See, for example, Section 12.2 in Schilling and Partzsch (2014) where the formula is derived from Lévy’s triple law.

The p -value for a test of H_0 equals $1 - F_S(\hat{T}_n)$, and since the series in (31) converges at an exponential rate, the p -value can be computed with high numerical precision by truncating the series. Given a significance level $\alpha \in (0, 1)$, we also let $z_{1-\alpha}$ denote the $(1 - \alpha)$ -quantile of F_S , which exists and is unique since the right-hand side of (31) is strictly increasing and continuous. The *Local Covariance Test* (LCT) with significance level α is then defined by

$$(32) \quad \Psi_n = \Psi_n^\alpha = \mathbb{1}(F_S(\hat{T}_n) > 1 - \alpha) = \mathbb{1}(\hat{T}_n > z_{1-\alpha}).$$

From Theorem 4.6, we can now deduce the asymptotic properties of the LCT under the hypothesis of conditional local independence. Recall that $\xrightarrow{D/\Theta_0}$ denotes uniform convergence in distribution under H_0 .

THEOREM 5.1. *Let Assumptions 4.1, 4.2 and 5.1 hold true. Then it holds that*

$$\hat{T}_n^\theta \xrightarrow{D/\Theta_0} S$$

as $n \rightarrow \infty$. As a consequence, for any $\alpha \in (0, 1)$,

$$\limsup_{n \rightarrow \infty} \sup_{\theta \in \Theta_0} \mathbb{P}(\Psi_n^{\alpha, \theta} = 1) \leq \alpha.$$

In other words, the local covariance test defined in (32) has uniform asymptotic level α .

In general, we cannot expect that the test has power against alternatives to H_0 for which the LCM is the zero function. This is analogous to other types of conditional independence tests based on conditional covariances, for example, GCM (Shah and Peters (2020)). However, we do have the following result that establishes power against local alternatives with $\|\gamma\|_\infty$ decaying at an order of at most $|J_n|^{-1/2}$.

THEOREM 5.2. *Let Assumptions 4.1 and 4.2 hold true. Using the additive residual process it holds that for any $0 < \alpha < \beta < 1$, there exists $c > 0$ such that*

$$\liminf_{n \rightarrow \infty} \inf_{\theta \in \mathcal{A}_{c,n}} \mathbb{P}(\Psi_n^{\alpha, \theta} = 1) \geq \beta,$$

where $\mathcal{A}_{c,n} = \{\theta \in \Theta \mid \|\gamma^\theta\|_\infty \geq c|J_n|^{-1/2}\}$.

5.2. *Extension to cross-fitting.* In Section 4, we considered sample splitting with observations indexed by J_n^c used to estimate the two models and with observations indexed by J_n used to estimate γ . Following Chernozhukov et al. (2018), we can improve efficiency by cross-fitting, that is, by flipping the roles of J_n and J_n^c to obtain a second equivalent estimator of γ . Heuristically, the two estimators are approximately independent, and thus their average should be a more efficient estimator. This procedure generalizes directly to a partition $J_n^1 \cup \dots \cup J_n^K = \{1, \dots, n\}$ of the indices into K disjoint folds. The partition is assumed to have a uniform asymptotic density, meaning that $|J_n^k|/n \rightarrow \frac{1}{K}$ as $n \rightarrow \infty$ for each k .

We estimate G and λ using $(J_n^k)^c = \{1, \dots, n\} \setminus J_n^k$ and subsequently estimate γ using J_n^k . Then the K -fold *Cross-fitted LCM estimator*, abbreviated as X-LCM, is defined as the average LCM estimator over the K folds, that is,

$$(33) \quad \check{\gamma}_t^{K,(n)} = \frac{1}{K} \sum_{k=1}^K \frac{1}{|J_n^k|} \sum_{j \in J_n^k} \int_0^t \hat{G}_{j,s}^{k,(n)} d\hat{M}_{j,s}^{k,(n)},$$

where for each $j \in J_n^k$, the processes $\hat{G}_j^{k,(n)}$ and $\hat{M}_j^{k,(n)}$ are the model predictions of G_j and M_j , respectively, based on training data indexed by $(J_n^k)^c$. We also define a K -fold version of the variance estimator:

$$(34) \quad \check{\gamma}_n^K(t) = \frac{1}{K} \sum_{k=1}^K \frac{1}{|J_n^k|} \sum_{j \in J_n^k} \int_0^t (\hat{G}_{j,s}^{k,(n)})^2 dN_{j,s}.$$

Now, similar to the LCT statistic, the cross-fitted estimator can be used to construct a test statistic,

$$(35) \quad \check{T}_n^K = \sqrt{\frac{n}{\check{\gamma}_{K,n}(1)}} \sup_{0 \leq t \leq 1} |\check{\gamma}_t^{K,(n)}|,$$

from which we define the following test of conditional local independence.

DEFINITION 5.3. Let $\alpha \in (0, 1)$ and let \check{T}_n^K be the test statistic from (35). The K -fold *Cross-fitted Local Covariance Test* (X-LCT) with significance level α is defined by

$$\check{\Psi}_n^K = \mathbb{1}(F_S(\check{T}_n^K) > 1 - \alpha) = \mathbb{1}(\check{T}_n^K > z_{1-\alpha}),$$

where $z_{1-\alpha}$ is the $(1 - \alpha)$ -quantile of the distribution function F_S given in (31).

We provide a summary of the computation of the X-LCT in Algorithm 2. The asymptotic analysis of $\hat{\gamma}^{(n)}$ generalizes to $\check{\gamma}^{K,(n)}$, but we will refrain from restating all results for the K -fold cross-fitted estimator. For simplicity, we focus on the fact that the X-LCT has asymptotic level α .

THEOREM 5.4. Suppose that Assumption 4.2 is satisfied for every sample split $J_n^k \cup (J_n^k)^c, k = 1, \dots, K$. Under Assumptions 4.1 and 5.1, the X-LCT statistic satisfies

$$\check{T}_n^{K,\theta} \xrightarrow{\mathcal{D}/\Theta_0} S$$

for $n \rightarrow \infty$. In particular, the X-LCT has uniform asymptotic level α .

Note that cross-fitting recovers full efficiency in the sense that the scaling factor is \sqrt{n} rather than $\sqrt{|J_n|}$, which leads to a more powerful test. Moreover, the asymptotic distribution of \check{T}_n^K does not depend on the number of folds K , and any difference between various choices of K can thus be attributed to finite sample errors. Larger values of K will allocate more data to estimation of G and λ , which intuitively should be the harder estimation problem. Following Remark 3.1 in Chernozhukov et al. (2018), we believe that a default choice of $K = 4$ or $K = 5$ should be reasonable in practice.

Algorithm 2: K -fold cross-fitted local covariance test (X-LCT)

1 input: processes $(N_j, X_j, Z_j)_{j=1, \dots, n}$, partition $J_n^1 \cup \dots \cup J_n^K$ of indices into K folds;
2 options: historical regression methods for estimation of λ and G given N and Z ,
3 discrete time grid $\mathbb{T} \subset [0, 1]$, significance level $\alpha \in (0, 1)$;
4 begin
5 **for** $k = 1, \dots, K$ **do**
6 | apply Algorithm 1 on sample split $J_n^k \cup (J_n^k)^c$ to compute $\tilde{\gamma}^{k,(n)}$ on grid \mathbb{T} ;
7 | use equation (27) on sample split $J_n^k \cup (J_n^k)^c$ to compute $\tilde{\mathcal{V}}_{k,n}(1)$;
8 **end**
9 compute $\check{\gamma}^{K,(n)} = \frac{1}{K} \sum_{k=1}^K \tilde{\gamma}^{k,(n)}$ on grid \mathbb{T} ;
10 compute $\check{\mathcal{V}}_{K,n}(1) = \frac{1}{K} \sum_{k=1}^K \tilde{\mathcal{V}}_{k,n}(1)$;
11 compute the X-LCT statistic $\check{T}_n^K = \sqrt{n} \cdot \max_{t \in \mathbb{T}} |\check{\gamma}_t^{K,(n)}| / \sqrt{\check{\mathcal{V}}_{K,n}(1)}$;
12 compute p -value $\check{p} = 1 - F_S(\check{T}_n^K)$ by truncating the series in equation (31).
13 end
14 output: the X-LCT $\check{\Psi}_n^K = \mathbb{1}(\check{p} < \alpha)$, and optionally the p -value \check{p} ;

6. Simulation study. In this section, we present the results from a simulation study based on the Cox example introduced in Section 2.2. We elaborate in Section 6.1 on the full model specification used for the simulation study, which will also illuminate how Π and λ can be modeled and estimated. The results from the simulation study focus on the distribution of the X-LCT statistic \check{T}_n^K and validate the asymptotic level and power of the X-LCT $\check{\Psi}_n^K$. The latter is also compared to a hazard ratio test based on the marginal Cox model (11). The simulations were implemented in Python and the code is available.¹

6.1. *Cox model continued.* Consider the same setup as in Section 2.2. To fully specify the model, we need to specify the distribution of the processes X, Y and Z . We suppose that X and Y can be written in terms of Z as

$$(36) \quad X_t = \int_0^t Z_s \rho_X(s, t) ds + V_t \quad \text{and} \quad Y_t = \int_0^t Z_s \rho_Y(s, t) ds + W_t,$$

where ρ_X and ρ_Y are two functions defined on the triangle $\{(s, t) \in [0, 1]^2 | s \leq t\}$, and where $V = (V_t)_{0 \leq t \leq 1}$ and $W = (W_t)_{0 \leq t \leq 1}$ are two noise processes with mean zero. The processes Z, V and W are assumed independent, which implies (10), and thus that N is conditionally locally independent of X given $\mathcal{F}_t = \mathcal{F}_t^{N,Z}$.

The specific dependency of X and Y on Z is known as the *historical functional linear model* in functional data analysis (Malfait and Ramsay (2003)). Within this model,

$$(37) \quad \Pi_t = \mathbb{E}(X_t | \mathcal{F}_{t-}) = \int_0^t Z_s \rho_X(s, t) ds,$$

and on $(T \geq t)$,

$$\mathbb{E}(e^{Y_t} | \mathcal{F}_t) = e^{\int_0^t Z_s \rho_Y(s, t) ds} \mathbb{E}(e^{W_t} | T \geq t) = e^{\tilde{\beta}_0(t) + \int_0^t Z_s \rho_Y(s, t) ds},$$

¹<https://github.com/AlexanderChristgau/nonparametric-cli-test>.

where $\tilde{\beta}_0(t) = \log(\mathbb{E}(e^{W_t} | T \geq t))$. Since $\lambda_t = \mathbb{1}(T \geq t)\lambda_t^0 e^{\beta Z_t} \mathbb{E}(e^{Y_t} | \mathcal{F}_t)$, it follows that on $(T \geq t)$,

$$(38) \quad \log(\lambda_t) = \beta_0(t) + \beta Z_t + \int_0^t Z_s \rho_Y(s, t) ds,$$

where the two baseline terms depending only on time have been merged into β_0 .

The computations above suggest how the estimators $\hat{\lambda}^{(n)}$ and $\hat{\Pi}^{(n)}$ could be constructed. That is, $\hat{\lambda}^{(n)}$ could be based on estimates of β , β_0 and ρ_Y from the observations $(T_j, Z_j)_{j \in J_n^c}$, and $\hat{\Pi}^{(n)}$ could be based on estimates of ρ_X from $(X_j, Z_j)_{j \in J_n^c}$. We would then have

$$\hat{\Pi}_{j,t}^{(n)} = \int_0^t Z_{j,s} \hat{\rho}_X^{(n)}(s, t) ds$$

for $j \in J_n$ where $\hat{\rho}_X^{(n)}$ denotes the estimate of ρ_X , and similarly for $\hat{\lambda}_{j,t}^{(n)}$. Particular choices of estimators $\hat{\rho}_X^{(n)}$ and $\hat{\rho}_Y^{(n)}$ and their theoretical properties are reviewed in Section D in the supplement. Our conclusion from this review is that for the historical functional linear model, sufficient rate results should be possible but have not yet been established rigorously.

6.2. Sampling scheme. The actual time-discretized simulations and computations were implemented using an equidistant grid $\mathbb{T} = (t_i)_{i=1}^q$ with $q = 128$ time points $0 = t_1 < \dots < t_q = 1$. Inspired by Harezlak et al. (2007), we generated the processes as follows: let $\xi \in \mathbb{R}^3$ and $V, W, \mathbb{W} \in \mathbb{R}^{\mathbb{T}}$ be independent random variables such that $\xi \sim \mathcal{N}(0, \mathbf{I}_3)$ and such that V, W and \mathbb{W} are identically distributed with $V_{t_1}, V_{t_2} - V_{t_1}, \dots, V_{t_q} - V_{t_{q-1}} \stackrel{\text{i.i.d.}}{\sim} \mathcal{N}(0, 1/q)$. Then the process Z is determined by $Z_t = \xi_1 + \xi_2 t + \sin(2\pi \xi_3 t) + \mathbb{W}_t$ for $t \in \mathbb{T}$. The processes X and Y were then given by the historical linear model (36) with kernels ρ_X and ρ_Y being one of the following four kernels:

$$\begin{aligned} \text{zero: } (s, t) &\mapsto 0, & \text{constant: } (s, t) &\mapsto 1, \\ \text{Gaussian: } (s, t) &\mapsto e^{-2(t-s)^2}, & \text{sine: } (s, t) &\mapsto \sin(4t - 20s). \end{aligned}$$

To compute X and Y , we evaluated the kernels on $\{(s, t) \in \mathbb{T}^2 | s \leq t\}$ and approximated the integrals by Riemann sums. The full intensity for $N_t = \mathbb{1}(T \leq t)$ was specified with a Weibull baseline of the form $\lambda_t^{\text{full}} = \mathbb{1}(T \geq t)\beta_1 t^2 \exp(\beta_2 Z_t + Y_t)$, for $\beta_1 > 0$ and a choice of $\beta_2 \in \{-1, 1\}$. To sample T , we applied the inverse hazard method, which utilizes that Λ_T^{full} is standard exponentially distributed. That is, we sampled $E \sim \text{Exp}(1)$ and numerically computed $T = \max\{t \in \mathbb{T} | \Lambda_t^{\text{full}} < E\}$ as a discretized approximation. For any given parameter setting, the baseline coefficient β_1 was chosen sufficiently large to ensure that $\Lambda_t^{\text{full}} \geq E$ would occur before time $t = 1$ in more than $\frac{q-1}{q} \cdot n$ samples.

The simulation setting used to sample the data for Figures 3 and 4 was $\beta_2 = -1$ and $\rho_X = \rho_Y = \text{constant}$.

With this setup, Assumption 4.1 is satisfied if V, W and \mathbb{W} were bounded. Since we use the Gaussian distribution, they are technically not bounded, but they could be made bounded by introducing a lower and upper cap. Due to the light tails of the Gaussian distribution such caps would have no noticeable effect on the simulation results, and the results we report are generated without a cap.

The implementation details for the X-LCT and the hazard ratio test are given in Section G.1 in the supplement.

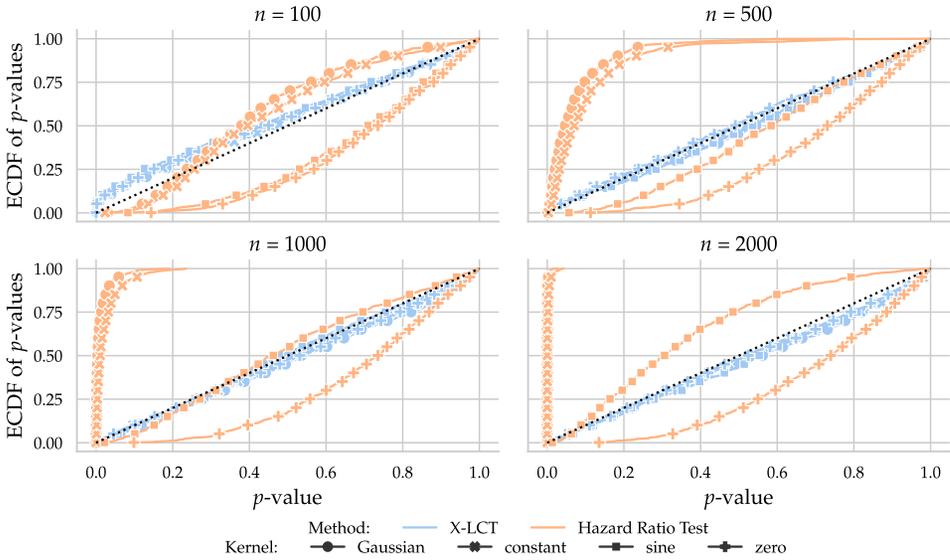


FIG. 5. Empirical cumulative distribution functions of simulated p -values for the cross-fitted local covariance test and the hazard ratio test. The simulated data satisfies the hypothesis of conditional local independence, so the p -values are supposed to be uniformly distributed, and the CDF should fall on the diagonal dotted line.

6.3. *Distributions of p -values under H_0 .* We examine the distributional approximation $\check{T}_n^K \stackrel{\text{as}}{\sim} S$ (cf. Theorem 5.4) by comparing the p -values $1 - F_S(\check{T}_n^K)$ to a uniform distribution. Figure 5 shows the empirical distribution functions of the p -values computed from data simulated according to the scheme described in the previous section. The results are aggregated over the two choices of $\beta_2 \in \{-1, 1\}$ since these two settings were found to be similar. For more detailed results from the experiment see Figure G.1 in Section G in the supplement, which also includes the p -values corresponding to the endpoint test statistic.

For the hazard ratio test, Figure 5 shows that the p -values are subuniform for the zero kernel. In this case, the marginal Cox model is correct, and the nonuniformity of the p -values can be explained by the L_2 -penalization. For the constant and Gaussian kernels, the hazard ratio test fails completely, whereas for the sine kernel, the mediated effect of Z on T through Y is more subtle, and the model misspecification only becomes apparent for $n = 2000$. Overall, these results are consistent with the reasoning in the Section 2.2: a test based on the misspecified Cox model will wrongly reject the hypothesis of conditional local independence.

For the proposed X-LCT, Figure 5 shows that the associated p -values are slightly anti-conservative for $n = 100$. This is to be expected, and can be explained by the finite sample errors leading to more extreme values of \check{T}_n^K than the approximation by S . As n increases, these errors become smaller, and for $n = 2000$ the p -values actually seem to be subuniform. The subuniformity may be explained by the time discretization, since the maximum of the process is taken over \mathbb{T} rather than $[0, 1]$. Figure G.3 in Section G in the supplement illustrates the asymptotic effect of the time discretization, which supports this claim. Another support of this claim is that the endpoint test does not appear to give subuniform p -values for large n ; see Figure G.1. We finally note that the distributions of the p -values for our proposed test is largely unaffected by the kernel used to generate the data.

6.4. *Power against local alternatives.* To investigate the power of the X-LCT, we construct local alternatives to H_0 in accordance with the right graph in Figure 2 by replacing Y_t by the process $Y_t + \frac{\rho_0}{\sqrt{n}}X_t$. That is, for $\rho_0 \neq 0$, blood pressure is then directly affected by pension savings, and N_t is no longer conditionally locally independent of X_t given \mathcal{F}_t . In

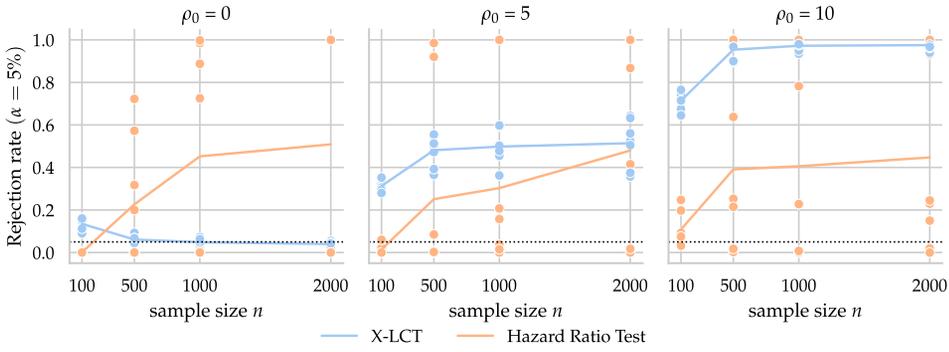


FIG. 6. For each $\rho_0 \in \{0, 5, 10\}$, the lines show the average rejection rates of our proposed test X-LCT (blue) and the hazard ratio test (orange) as functions of sample size, with each average taken over 8 different settings. For each setting, the rejection rate is computed from 400 simulated data sets at a 5% significance level and the rejection rate is displayed with a dot.

terms of the full intensity, these local alternatives are equivalent to

$$(39) \quad \lambda_t^{\text{full}} = \mathbb{1}(T \geq t) \beta_1 t^2 \exp\left(\beta_2 Z_t + Y_t + \frac{\rho_0}{\sqrt{n}} X_t\right).$$

We simulated data for the dependency parameter $\rho_0 \in \{0, 5, 10\}$. Note that $\rho_0 = 0$ corresponds to our previous sampling scheme with conditional local independence. For each of the $96 = 4 \times 2 \times 4 \times 3$ choices of kernel, β_2 , n and ρ_0 , we ran the tests 400 times and computed the p -values. For simplicity, we report the rejection rate at an $\alpha = 5\%$ significance level and the results are shown in Figure 6.

In the leftmost panel, the data was generated under H_0 and the plot shows what we noted previously, namely that the X-LCT holds level for large n , whereas the hazard ratio test does not. For the local alternatives, $\rho_0 = 5$ and $\rho_0 = 10$, we note that the power of the hazard ratio test is quite sensitive to the simulation settings. For some settings it has no power, while for others it has some power.

In contrast, the proposed X-LCT has power against all of the local alternatives. The power increases with n initially but stabilizes from around $n = 1000$. This is similar to the behavior observed under the null hypothesis and is not surprising. We expect that the sample size needs to be sufficiently large for the nonparametric estimators to work sufficiently well, and we expect the sufficient sample size to be mostly unaffected by the value of ρ_0 . For fixed n , we also note that the power of $\check{\Psi}_n^K$ is fairly robust with respect to the choice of β_2 and the choice of kernel. Overall, we find that the X-LCT is applicable in these settings with historical effects; it has consistent power against the \sqrt{n} alternatives while controlling type I error for n reasonably large.

In Section G.2 in the supplement, we provide additional numerical results for time-varying alternatives, and we compare the X-LCT with its endpoint counterpart.

7. Discussion. The LCM was introduced as a functional parameter that quantifies deviations from the hypothesis H_0 of conditional local independence. We showed how the parameter may be expressed in several ways, but that it is the representation in terms of the residual process that allows us to estimate the LCM with a \sqrt{n} -rate under H_0 without parametric model assumptions. The residual process was introduced as an abstract model of X_t for each t given the history up to time t , and we showed that such a residualization could be viewed as a form of orthogonalization. Similar ideas have been used recently for classical conditional independence testing, such as GCM (Shah and Peters (2020)), tests based on the partial copula (Petersen and Hansen (2021)) and GHCM (Lundborg, Shah and Peters

(2022)). It is, however, not possible to use any of these to test H_0 , which cannot be expressed as a classical conditional independence. Our test based on the LCM is the first nonparametric test of conditional local independence with substantial theoretical support, and we propose to test H_0 in practice by using X-LCT based on the cross-fitted estimator of LCM.

Contrary to the tests of conditional independence mentioned above, we need sample splitting—even under H_0 —to achieve our asymptotic results. We do not believe that this can be avoided. The standard argument to avoid this uses classical conditional independence in a crucial way, which does not translate into our framework—basically because we condition on information that changes with time. Our simulation study also indicates that sample splitting or cross-fitting is needed in practice for the LCM estimator to be unbiased under H_0 .

While our cross-fitted estimator of the LCM, the X-LCM, share some of the general patterns of other double machine learning procedures—including the overall decomposition (17)—our analysis and results required a range of generalizations of known results and some novel ideas. The asymptotic distribution of the leading term, $U^{(n)}$, is also a well-known consequence of Rebolledo’s CLT; see, for example, Section V.4 in (Andersen et al. (1993)) for related results in the context of survival analysis. However, we generalized this result to uniform convergence in the Skorokhod space $D[0, 1]$, and we introduced new techniques for handling the remainder terms. These novel techniques are made necessary by the decomposition (17) being a decomposition of stochastic processes indexed by time. We outline below the three most important technical contributions we made.

First, to obtain uniform control of level and power, all asymptotic results in Section 4 are formulated in terms of uniform stochastic convergence. Since this notion of convergence had not previously been considered on general metric spaces, and especially not on the Skorokhod space, we had to develop the necessary theory. This development could be of independent interest, and we have collected the general definitions and main results on uniform stochastic convergence in metric spaces in Section B in the supplement. This framework also allowed us to show a uniform version of Rebolledo’s martingale CLT in Section C in the supplement.

Second, to establish distributional convergence under H_0 , we need to control the remainder terms $R_{i,t}^{(n)}$ uniformly over t . The third term, $R_3^{(n)}$, is simple to bound, and by exploiting Doob’s submartingale inequality, the second term, $R_2^{(n)}$, can also be bounded. The most difficult first term, $R_1^{(n)}$, was controlled using stochastic equicontinuity via an exponential tail bound and the use of the chaining lemma. The necessary general uniform stochastic equicontinuity and chaining arguments are collected in Section C.3 in the supplement.

Third, to achieve rate results in the alternative, the processes $D_1^{(n)}$ and $D_2^{(n)}$ must be controlled. The process $D_1^{(n)}$ does, like $U^{(n)}$, not involve any estimation, and its distributional convergence follows from a general CLT argument for continuous stochastic processes. The term $D_2^{(n)}$ is more difficult to handle, as it may not have mean zero if G_t is not the additive residual process. However, X_t cancels out in $\hat{G}_t^{(n)} - G_t$ for the additive residual process, which makes the difference \mathcal{F}_t -predictable, and $D_2^{(n)}$ can then be bounded similar to $R_1^{(n)}$. For a general residual process, it seems possible for $D_2^{(n)}$ to have a bias of order $\sqrt{|J_n|}g(n)$.

Our main result, Theorem 4.6, is stated under two assumptions. The second, Assumption 4.2, is a straightforward generalization to our setup of similar assumptions in the double machine learning literature on rates of convergence for the two estimators used. Both estimation errors are measured using a 2-norm, and it is plausible that we can relax one norm to a weaker form of convergence if we simultaneously strengthen the other norm. The first assumption, Assumption 4.1, requires uniform bounds on both λ and G . This is a strong assumption but perhaps not particularly problematic from a practical viewpoint. Indeed, G is a process we can choose, and we can thus make it bounded if necessary. And though many

theoretically interesting counting process models have unbounded intensities, a large cap on the intensity will make no difference in practice. We believe, nevertheless, that it is possible to relax Assumption 4.1 to a weaker form of control on the magnitudes of λ and G as functions of time, for example, moment bounds uniform in θ . However, such a generalization will come at the expense of considerably more technical proofs, and we did not pursue this line of research.

A major practical question is whether we can estimate λ and G with sufficient rates, for example, $n^{-\frac{1}{4}+\varepsilon}$. In Section D in the supplement, we give an overview of some known and some conjectured rate results for specific forms of λ and Π . Beyond parametric models, we conclude that the existing rate results are scarce, and we regard it as an independent research project to establish rates for general historical regression methods.

Another question is whether we can replace the counting process N by a more general semimartingale. Commenges and Gégout-Petit (2009) define conditional local independence for a class of special semimartingales, and Mogensen, Malinsky and Hansen (2018) and Mogensen and Hansen (2022) show global Markov properties for local independence graphs of certain Itô processes, which are, in particular, special semimartingales. Thus, conditional local independence is well-defined beyond counting processes, and we believe that most definitions and results of this paper would generalize beyond N being a counting process. Besides some additional technical challenges, the major practical obstacle with such a generalization is that we cannot realistically assume to have completely observed sample paths of Itô processes, say. The discrete time nature of the observations should then be included in the analysis, and this is beyond the scope of the present paper.

Irrespective of the remaining open problems, the simulation study demonstrated some important properties of our proposed test, the X-LCT. First, it was fairly simple to implement for the specific example considered using some standard estimation techniques that were not tailored to the specific model class. Second, it had good level and power properties and clearly outperformed the test based on the misspecified marginal Cox model. Third, both Neyman orthogonalization as well as cross-fitting were pivotal for achieving the good properties of the test.

Acknowledgments. The authors would like to thank the Associate Editor and the anonymous reviewers for constructive comments that lead to substantial improvements of the paper.

Funding. The work was supported by Novo Nordisk Foundation Grant NNF20OC0062897.

SUPPLEMENTARY MATERIAL

Supplementary article (DOI: [10.1214/23-AOS2323SUPPA](https://doi.org/10.1214/23-AOS2323SUPPA); .pdf). The supplementary article (Christgau, Petersen and Hansen (2023)) contains proofs of results from the main text, auxiliary results, additional discussions and figures.

Computer code (DOI: [10.1214/23-AOS2323SUPPB](https://doi.org/10.1214/23-AOS2323SUPPB); .zip). Computer code that reproduces numerical results for the simulation study in Section 6.

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SUPPLEMENT TO ‘NONPARAMETRIC CONDITIONAL LOCAL INDEPENDENCE TESTING’

BY ALEXANDER MANGULAD CHRISTGAU, LASSE PETERSEN, AND NIELS RICHARD HANSEN

This is the supplementary material for [Christgau, Petersen and Hansen \(2023\)](#), hereafter referred to as the main text.

In Section [A](#), we give the proofs of the results of the main text. In Section [B](#), we formulate a general uniform asymptotic theory for metric spaces, whereafter we specialize the theory to the Skorokhod space $D[0, 1]$ and chaining of stochastic processes. In Section [C](#), we state Rebolledo’s martingale central limit theorem, and then we generalize the result to a uniform version that is used in the proofs. In Section [D](#), we discuss estimation of the intensity λ and the residual process G in practice. In particular, we compare known rate results with the rates required in Assumption [4.2](#). In Section [E](#), we compare the LCM estimator with existing work in semiparametric survival models. In Section [F](#), we provide mathematical details regarding Neyman orthogonality. Finally, Section [G](#) contains additional details, numerical results, and figures related to the simulation study of Section [6](#).

A. Proofs of results in the main text.

A.1. *Proof of Proposition [2.5](#).* The process G_t is càglàd and \mathcal{G}_t -predictable by assumption, and the process $I = (I_t)$ is a stochastic integral of G_t w.r.t. a local \mathcal{G}_t -martingale under the hypothesis H_0 . It is thus also a local \mathcal{G}_t -martingale under H_0 . By definition, $I_0 = 0$, and if I is a martingale, $\gamma_t = \mathbb{E}(I_t) = \mathbb{E}(I_0) = 0$. \square

A.2. *Proof of proposition [2.6](#).* Suppose that H is non-negative, càglàd and \mathcal{G}_t -predictable, then since $\int_0^t H_s d\mathbf{M}_s$ is a local \mathcal{G}_t -martingale it follows by monotone convergence along a localizing sequence that

$$(1) \quad \mathbb{E} \left(\int_0^t H_s dN_s \right) = \mathbb{E} \left(\int_0^t H_s \lambda_s ds \right) = \int_0^t \mathbb{E}(H_s \lambda_s) ds$$

for all $t \in [0, 1]$. We can apply the identity above with H the positive and negative part of G , respectively, and the integrability assumption ensures that (1) also holds with $H = G$. It follows that

$$\gamma_t = \mathbb{E}(I_t) = \mathbb{E} \left(\int_0^t G_s (\lambda_s - \lambda_s) ds \right) = \int_0^t \mathbb{E}(G_s (\lambda_s - \lambda_s)) ds.$$

The latter expectation is indeed a covariance since $\mathbb{E}(G_s) = \mathbb{E}(\mathbb{E}(G_s | \mathcal{F}_{s-})) = 0$. \square

A.3. *Proof of Lemma [4.1](#).* Before proving Lemma [4.1](#), we first state general martingale criteria in the context of counting processes.

LEMMA A.1. *Let (H_t) be a locally bounded \mathcal{G}_t -predictable process, let N be a counting process with a \mathcal{G}_t -intensity λ_t , and let $\mathbf{M}_t = N_t - \int_0^t \lambda_s ds$.*

If $\int_0^1 \lambda_s ds$ (or equivalently N_1) is integrable, then \mathbf{M}_t and $\mathbf{M}_t^2 - \int_0^t \lambda_s ds$ are each \mathcal{G}_t -martingales. If, in addition, $\int_0^1 H_s^2 \lambda_s ds$ is integrable, then $\int_0^t H_s d\mathbf{M}_s$ is a mean zero square integrable martingale.

PROOF. The first part is Lemma 2.3.2 and Theorem 2.5.3 in [Fleming and Harrington \(2011\)](#). For the second part, assume that $\int_0^1 \lambda_s ds$ and $\int_0^1 H_s^2 \lambda_s ds$ are both integrable. In this case, the \mathcal{G}_t -predictable quadratic variation of \mathbf{M} is $\langle \mathbf{M} \rangle(t) = \int_0^t \lambda_s ds$ by the first part. Then it remains to note that (H_t) is a locally bounded \mathcal{G}_t -predictable process, so the conditions of Theorem 2.4.4 in [Fleming and Harrington \(2011\)](#) are satisfied if $\int_0^1 H_s^2 \lambda_s ds$ is integrable. This establishes the second part. \square

We now return to the proof of Lemma 4.1. Let $f \in C(\mathbb{R})$, and we shall prove that $\int_0^t f(G_s) d\mathbf{M}_s$ is a mean zero, square integrable \mathcal{G}_t -martingale. The proof for the integral with $f(\hat{G}_s^{(n)})$ is identical.

Continuity of f implies that $C_f := \sup_{x \in [-C', C']} |f(x)| < \infty$ and that $(f(G_t))$ is a \mathcal{G}_t -predictable process. By Assumption 4.1, the process $(f(G_t))$ is almost surely bounded by C_f and therefore

$$\mathbb{E} \left(\int_0^1 f(G_s)^2 \lambda_s ds \right) \leq C_f^2 C < \infty.$$

Thus we can apply Lemma A.1 to conclude that $\int_0^t f(G_s) d\mathbf{M}_s$ is a mean zero, square integrable \mathcal{G}_t -martingale. \square

A.4. *Proof of Proposition 4.3.* As noted elsewhere, the explicit parametrization of all objects by θ is notationally heavy, and there will thus be an implicit parameter value $\theta \in \Theta$ in most of the subsequent constructions and arguments.

To simplify notation we write

$$U_t^{(n)} = \sum_{j \in J_n} \int_0^t H_{j,s}^{(n)} d\mathbf{M}_{j,s}, \quad \text{where} \quad H_{j,s}^{(n)} = \frac{G_{j,s}}{\sqrt{|J_n|}}.$$

We will use a uniform extension of Rebolledo's martingale central limit theorem on the sequence $(U^{(n)})_{n \geq 1}$ to show the result. See Section C for a discussion of Rebolledo's CLT and Theorem C.4 for its uniform extension.

Define $\tilde{\mathcal{G}}_t^n$ be the smallest right continuous and complete filtration generated by the filtrations $\{\mathcal{G}_{j,t} \mid j \in J_n\}$. We can apply Lemma 4.1 to each of the terms of $U^{(n)}$ to conclude that the j -th term is a square integrable, mean zero $\mathcal{G}_{j,t}^n$ -martingale. By independence of the observations for each j , we can enlarge the filtration for each term and conclude that they are also square integrable, mean zero $\tilde{\mathcal{G}}_t^n$ -martingales. Thus $U^{(n)}$ is also a square integrable, mean zero $\tilde{\mathcal{G}}_t^n$ -martingale.

To apply Theorem C.4 first establish that the conditions in Equation (11) are fulfilled. By Proposition C.5, we have that

$$\langle U^{(n)} \rangle(t) = \sum_{j \in J_n} \int_0^t (H_{j,s}^{(n)})^2 \lambda_{j,s} ds = \frac{1}{|J_n|} \sum_{j \in J_n} \int_0^t G_{j,s}^2 \lambda_{j,s} ds.$$

By directly applying the bounds from Assumption 4.1, we see that the square mean of $\int_0^t G_s^2 \lambda_s ds$ is bounded by $C^2(C')^4$. Thus, for fixed $t \in [0, 1]$, the uniform law of large numbers ([Shah and Peters, 2020](#), Lemma 19) gives that

$$\langle U^{(n)} \rangle(t) = \frac{1}{|J_n|} \sum_{j \in J_n} \int_0^t G_{j,s}^2 \lambda_{j,s} ds \xrightarrow{P/\Theta} \mathbb{E} \left(\int_0^t G_s^2 \lambda_s ds \right) = \mathcal{V}(t)$$

for $n \rightarrow \infty$, since the integrals are i.i.d. with the same distribution as $\int_0^t G_s^2 \lambda_s ds$. This establishes the first part of the condition in Equation (11). For the second part, we also have from Proposition C.5 that

$$(2) \quad \begin{aligned} \langle U_\varepsilon^{(n)} \rangle(t) &= \sum_{j \in J_n} \int_0^t \left(H_{j,s}^{(n)} \right)^2 \mathbb{1} \left(|H_{j,s}^{(n)}| \geq \varepsilon \right) d\Lambda_{j,s} \\ &= \frac{1}{|J_n|} \sum_{j \in J_n} \int_0^t G_{j,s}^2 \mathbb{1} \left(|G_{j,s}| \geq \varepsilon \sqrt{|J_n|} \right) \lambda_{j,s} ds \end{aligned}$$

for each $t \in [0, 1]$ and $\varepsilon > 0$. From Assumption 4.1, we note that for n sufficiently large such that $|J_n| > (C')^2 / \varepsilon^2$, it holds that $\mathbb{P} \left(|G_{j,s}| \geq \varepsilon \sqrt{|J_n|} \right) = 0$ for all $j \in J_n$. As a consequence, the terms in (2) are almost surely zero for n sufficiently large uniformly over Θ . It follows that $\langle U_\varepsilon^{(n)} \rangle(t) \xrightarrow{P/\Theta} 0$, which establishes the second part of (11).

We finally note that the collection of variance functions, $(\mathcal{V}^\theta)_{\theta \in \Theta}$, is uniformly equicontinuous and bounded above under Assumption 4.1. This is established in Lemma A.2 below. We have thus verified all the conditions of Theorem C.4, so we conclude that

$$U^{(n),\theta} \xrightarrow{D/\Theta} U^\theta$$

in $D[0, 1]$ as $n \rightarrow \infty$, where U^θ is a mean zero continuous Gaussian martingale with variance function \mathcal{V}^θ . \square

Note that the convergence of (2) is established directly from the uniform bounds in Assumption 4.1. However, the convergence could also be established under a milder conditions with alternative arguments. For example, under the weaker assumption of uniformly bounded variance functions, dominated convergence can be used to establish L_1 -convergence.

In the proof above we invoked the following lemma, which we will also use in several proofs in the sequel.

LEMMA A.2. *Under Assumption 4.1, the collections $(\gamma^\theta)_{\theta \in \Theta}$ and $(\mathcal{V}^\theta)_{\theta \in \Theta}$ are each uniformly Lipschitz and in particular uniformly equicontinuous. Moreover, it holds almost surely that*

$$\sup_{t \in [0,1]} |\gamma_t| \leq 2CC' \quad \text{and} \quad \mathcal{V}(1) = \mathbb{E} \left(\int_0^1 G_s^2 \lambda_s ds \right) \leq C(C')^2.$$

PROOF. For any $0 \leq s < t \leq 1$, a direct application of Assumption 4.1 and Proposition 4.1 yields

$$|\gamma_t - \gamma_s| \leq \mathbb{E} \left| \int_s^t G_u dN_u - \int_s^t G_u \lambda_u du \right| \leq \mathbb{E} \left(\int_s^t |G_u| (\lambda_u + \lambda_u) du \right) \leq 2CC'(t-s),$$

and similarly,

$$\mathcal{V}(t) - \mathcal{V}(s) = \mathbb{E} \left(\int_s^t G_u^2 \lambda_u du \right) \leq C(C')^2(t-s).$$

This establishes the first part. The bounds follow from inserting $(s, t) = (0, t)$ in the first inequality and $(s, t) = (0, 1)$ in the second inequality. \square

A.5. *Proof of Proposition 4.4.* We will divide the proof into three lemmas for each of the remainder terms $R_1^{(n)}$, $R_2^{(n)}$ and $R_3^{(n)}$, where we establish convergence to the zero-process uniformly over t and θ . However, note that the notion of uniform convergence differs for the process index, $t \in [0, 1]$, and the parameter, $\theta \in \Theta$, as we need to show that

$$\forall i \in \{1, 2, 3\} \forall \varepsilon > 0: \quad \lim_{n \rightarrow \infty} \sup_{\theta \in \Theta} \mathbb{P} \left(\sup_{t \in [0, 1]} |R_{i,t}^\theta| > \varepsilon \right) = 0.$$

For a general discussion of the relation between weak convergence and convergence in probability uniformly as a stochastic process, see [Newey \(1991\)](#). For a general discussion of uniform stochastic convergence over a distribution parameter, see [Section B](#) and the references contained therein. In [Section B.3](#), we discuss the combination of both convergences.

As in the proof of [Proposition 4.3](#), $\tilde{\mathcal{G}}_t^n$ denotes the smallest right continuous and complete filtration generated by the filtrations $\{\mathcal{G}_{j,t} \mid j \in J_n\}$. Analogously, we let $\tilde{\mathcal{G}}_t^{n,c}$ be the smallest right continuous and complete filtration generated by the filtrations $\{\mathcal{G}_{j,t}^c \mid j \in J_n^c\}$. We start by considering $R_3^{(n)}$, since this is the easiest case.

LEMMA A.3. *Under Assumption 4.2 it holds that $\sup_{t \in [0, 1]} |R_{3,t}^{(n)}| \xrightarrow{P/\Theta} 0$.*

PROOF. We will show the result by showing that

$$\sup_{\theta \in \Theta} \mathbb{E} \left(\sup_{0 \leq t \leq 1} |R_{3,t}^{(n), \theta}| \right) \rightarrow 0$$

as $n \rightarrow \infty$. Using that the random variables

$$\sup_{0 \leq t \leq 1} |G_{j,t} - \hat{G}_{j,t}^{(n)}| \cdot \sup_{0 \leq t \leq 1} |\lambda_{j,t} - \hat{\lambda}_{j,t}^{(n)}|$$

for $j \in J_n$ are identically distributed for each fixed $n \geq 2$, we have that

$$\begin{aligned} & \mathbb{E} \left(\sup_{0 \leq t \leq 1} |R_{3,t}^{(n)}| \right) \\ &= \mathbb{E} \left(\sup_{0 \leq t \leq 1} \left| \frac{1}{\sqrt{|J_n|}} \sum_{j \in J_n} \int_0^t (G_{j,s} - \hat{G}_{j,s}^{(n)}) (\lambda_{j,s} - \hat{\lambda}_{j,s}^{(n)}) ds \right| \right) \\ &\leq \frac{1}{\sqrt{|J_n|}} \sum_{j \in J_n} \mathbb{E} \left(\sup_{0 \leq t \leq 1} \int_0^t |G_{j,s} - \hat{G}_{j,s}^{(n)}| \cdot |\lambda_{j,s} - \hat{\lambda}_{j,s}^{(n)}| ds \right) \\ &= \sqrt{|J_n|} \mathbb{E} \left(\int_0^1 |G_s - \hat{G}_s^{(n)}| \cdot |\lambda_s - \hat{\lambda}_s^{(n)}| ds \right) \\ &\leq \sqrt{|J_n|} \mathbb{E} \left(\sqrt{\int_0^1 (G_s - \hat{G}_s^{(n)})^2 ds} \sqrt{\int_0^1 (\lambda_s - \hat{\lambda}_s^{(n)})^2 ds} \right) \\ &\leq \sqrt{|J_n|} \sqrt{\mathbb{E} \left(\int_0^1 (G_s - \hat{G}_s^{(n)})^2 ds \right)} \sqrt{\mathbb{E} \left(\int_0^1 (\lambda_s - \hat{\lambda}_s^{(n)})^2 ds \right)} \\ &= \sqrt{|J_n|} g(n) h(n). \end{aligned}$$

By [Assumption 4.2](#), $\sqrt{|J_n|} g(n) h(n) \rightarrow 0$ uniformly over Θ as $n \rightarrow \infty$, so the result follows. \square

Next we proceed to the remainder process $R_2^{(n)}$.

LEMMA A.4. *Under Assumptions 4.1 and 4.2, it holds that $\sup_{t \in [0,1]} |R_{2,t}^{(n)}| \xrightarrow{P/\Theta} 0$.*

PROOF. We first write

$$R_{2,t}^{(n)} = \frac{1}{\sqrt{|J_n|}} \sum_{j \in J_n} \int_0^t (G_{j,s} - \hat{G}_{j,s}^{(n)}) d\mathbf{M}_{j,s},$$

and note that $R_{2,t}^{(n)}$ is a square integrable, mean zero $\tilde{\mathcal{G}}_t^n$ -martingale conditionally on $\tilde{\mathcal{G}}_1^{n,c}$. This follows by applying Lemma 4.1 to each of the terms, which are i.i.d. conditionally on $\tilde{\mathcal{G}}_1^{n,c}$. We conclude that the squared process $(R_{2,t}^{(n)})^2$ is a $\tilde{\mathcal{G}}_t^n$ -submartingale conditionally on $\tilde{\mathcal{G}}_1^{n,c}$. By Doob's submartingale inequality we have that

$$\begin{aligned} \mathbb{P} \left(\sup_{0 \leq t \leq 1} |R_{2,t}^{(n)}| \geq \varepsilon \right) &= \mathbb{P} \left(\sup_{0 \leq t \leq 1} (R_{2,t}^{(n)})^2 \geq \varepsilon^2 \right) \\ &= \mathbb{E} \left(\mathbb{P} \left(\sup_{0 \leq t \leq 1} (R_{2,t}^{(n)})^2 \geq \varepsilon^2 \mid \tilde{\mathcal{G}}_1^{n,c} \right) \right) \\ &\leq \frac{\mathbb{E} \left(\text{Var} \left(R_{2,1}^{(n)} \mid \tilde{\mathcal{G}}_1^{n,c} \right) \right)}{\varepsilon^2} \end{aligned}$$

for $\varepsilon > 0$. The collection of random variables

$$\left(\int_0^1 (G_{j,s} - \hat{G}_{j,s}^{(n)}) d\mathbf{M}_{j,s} \right)_{j \in J_n}$$

are i.i.d. conditionally on $\tilde{\mathcal{G}}_1^{n,c}$. Therefore,

$$\begin{aligned} \text{Var} \left(R_{2,1}^{(n)} \mid \tilde{\mathcal{G}}_1^{n,c} \right) &= \frac{1}{|J_n|} \sum_{j \in J_n} \text{Var} \left(\int_0^1 (G_{j,s} - \hat{G}_{j,s}^{(n)}) d\mathbf{M}_{j,s} \mid \tilde{\mathcal{G}}_1^{n,c} \right) \\ &= \mathbb{E} \left(\int_0^1 (G_s - \hat{G}_s^{(n)})^2 d\langle \mathbf{M} \rangle_s \mid \tilde{\mathcal{G}}_1^{n,c} \right) \\ &= \mathbb{E} \left(\int_0^1 (G_s - \hat{G}_s^{(n)})^2 \lambda_s ds \mid \tilde{\mathcal{G}}_1^{n,c} \right) \\ &\leq C \cdot \mathbb{E} \left(\int_0^1 (G_s - \hat{G}_s^{(n)})^2 ds \mid \tilde{\mathcal{G}}_1^{n,c} \right) \end{aligned}$$

where we have used that λ_t is bounded by Assumption 4.1 (i). Thus

$$\mathbb{E} \left(\text{Var} \left(R_{2,1}^{(n)} \mid \tilde{\mathcal{G}}_1^{n,c} \right) \right) \leq C \cdot \mathbb{E} \left(\int_0^1 (G_s - \hat{G}_s^{(n)})^2 ds \right) = C \cdot g(n)^2,$$

and we conclude that

$$\mathbb{P} \left(\sup_{0 \leq t \leq 1} |R_{2,t}^{(n)}| \geq \varepsilon \right) \leq \frac{C \cdot g(n)^2}{\varepsilon^2} \rightarrow 0,$$

as $n \rightarrow \infty$ uniformly over Θ by Assumption 4.2. \square

Before proving that $R_1^{(n)}$ converges weakly to the zero-process, we will need two auxiliary lemmas. The first is a conditional version of Hoeffding's lemma, which lets us conclude conditional sub-Gaussianity. Recall that a mean zero random variable A is sub-Gaussian with variance factor $\nu > 0$ if

$$\log \mathbb{E}(e^{xA}) \leq \frac{x^2 \nu}{2}$$

for all $x \in \mathbb{R}$. See, for example, [Boucheron, Lugosi and Massart \(2013\)](#), Lemma 2.2, for the classical unconditional version.

LEMMA A.5 (conditional Hoeffding's lemma). *Let Y be a random variable taking values on a bounded interval $[a, b]$, satisfying $\mathbb{E}[Y | \mathcal{G}] = 0$ for a σ -algebra \mathcal{G} .*

Then $\log \mathbb{E}(e^{xY} | \mathcal{G}) \leq (b-a)^2 x^2 / 8$ almost surely for all $x \in \mathbb{R}$.

PROOF. Fix $x \in \mathbb{R}$. By convexity of the exponential function we have

$$e^{xy} \leq \frac{b-y}{b-a} e^{xa} + \frac{y-a}{b-a} e^{xb}, \quad y \in [a, b].$$

Inserting Y in place of y and taking the conditional expectation yields

$$\mathbb{E}[e^{xY} | \mathcal{G}] \leq \frac{b}{b-a} e^{xa} - \frac{a}{b-a} e^{xb} = e^{L(x(b-a))}$$

almost surely, where $L(h) = \frac{ha}{b-a} + \log(1 + \frac{a-e^h a}{b-a})$. Standard calculations show that $L(0) = L'(0) = 0$, and the AM-GM inequality implies

$$L''(h) = -\frac{abe^h}{(b-ae^h)^2} \leq \frac{1}{4}.$$

Thus, a second order Taylor expansion yields that $L(h) \leq \frac{1}{8}h^2$, and it follows that $\log \mathbb{E}[e^{xY} | \mathcal{G}] \leq \frac{(b-a)^2}{8}x^2$ as desired. \square

For the next lemma define for $s, t \in [0, 1]$ with $s < t$

$$W^{s,t} = \frac{1}{t-s} \int_s^t G_u (\lambda_u - \hat{\lambda}_u^{(n)}) du.$$

LEMMA A.6. *Let Assumption 4.1 hold true. Then, for any $0 \leq s < t \leq 1$, it holds that $\mathbb{E}(W^{s,t} | \tilde{\mathcal{G}}_1^{n,c}) = 0$ and that $W^{s,t}$ is sub-Gaussian conditionally on $\tilde{\mathcal{G}}_1^{n,c}$ with variance factor $\nu = (2CC')^2$, that is,*

$$\log \mathbb{E}(e^{xW^{s,t}} | \tilde{\mathcal{G}}_1^{n,c}) \leq 2(xCC')^2$$

for all $s < t$ and $x \in \mathbb{R}$.

PROOF. For fixed $u \in [0, 1]$, note that

$$\begin{aligned} \mathbb{E} \left(G_u \left(\lambda_u - \hat{\lambda}_u^{(n)} \right) \mid \tilde{\mathcal{G}}_1^{n,c} \right) &= \mathbb{E} \left(\mathbb{E} \left(G_u \left(\lambda_u - \hat{\lambda}_u^{(n)} \right) \mid \mathcal{F}_{s-} \vee \tilde{\mathcal{G}}_1^{n,c} \right) \mid \tilde{\mathcal{G}}_1^{n,c} \right) \\ (3) \quad &= \mathbb{E} \left(\mathbb{E} \left(G_u \mid \mathcal{F}_{s-} \right) \left(\lambda_u - \hat{\lambda}_u^{(n)} \right) \mid \tilde{\mathcal{G}}_1^{n,c} \right) = 0, \end{aligned}$$

where we have used that $\lambda_t - \hat{\lambda}_t^{(n)}$ is \mathcal{F}_t -predictable conditionally on $\tilde{\mathcal{G}}_1^{n,c}$, that G_t is independent of $\tilde{\mathcal{G}}_1^{n,c}$ since it is \mathcal{G}_t -predictable, and that $\mathbb{E}(G_s | \mathcal{F}_{s-}) = 0$ per definition. By

applying the conditional Fubini theorem (Schilling, 2017, Theorem 27.17), we conclude that $\mathbb{E}(W^{s,t} | \tilde{\mathcal{G}}_1^{n,c}) = 0$.

We can now use the conditional version of Hoeffding's lemma formulated in Lemma A.5. Indeed, we have that for all $s < t$

$$\begin{aligned} |W^{s,t}| &\leq \frac{1}{t-s} \int_s^t |G_u| |(\lambda_u - \hat{\lambda}_u^{(n)})| du \\ &\leq \sup_{0 \leq u \leq 1} |G_u| \sup_{0 \leq u \leq 1} |(\lambda_u - \hat{\lambda}_u^{(n)})| \leq 2CC' \end{aligned}$$

by Assumption 4.1. Hence, for all $s < t$, Lemma A.5 lets us conclude that $\log \mathbb{E}(e^{xW^{s,t}} | \tilde{\mathcal{G}}_1^{n,c}) \leq 2(xCC')^2$, $x \in \mathbb{R}$. \square

Then we have the following regarding $R_1^{(n)}$.

LEMMA A.7. *Under Assumptions 4.2 and 4.1 it holds that $\sup_{t \in [0,1]} |R_{1,t}^{(n)}| \xrightarrow{P/\Theta} 0$.*

PROOF. The proof consists of two parts. First we show that for each $t \in [0, 1]$ it holds that

$$R_{1,t}^{(n)} \xrightarrow{P/\Theta} 0$$

for $n \rightarrow \infty$. Then we show *stochastic equicontinuity* of the process $R_1^{(n)}$ uniformly over Θ , and by Lemma B.16 it follows that

$$\sup_{t \in [0,1]} |R_{1,t}^{(n)}| \xrightarrow{P/\Theta} 0.$$

This is a direct generalization of Theorem 2.1 in Newey (1991). The collection of random variables

$$\left(G_{j,s} \left(\lambda_{j,s} - \hat{\lambda}_{j,s}^{(n)} \right) \right)_{j \in J_n}$$

are i.i.d. conditionally on $\tilde{\mathcal{G}}_1^{n,c}$. Therefore, an application of the conditional Fubini theorem yields

$$\mathbb{E}(R_{1,t} | \tilde{\mathcal{G}}_1^{n,c}) = \frac{1}{\sqrt{|J_n|}} \sum_{j \in J_n} \int_0^t \mathbb{E} \left(G_{j,s} \left(\lambda_{j,s} - \hat{\lambda}_{j,s}^{(n)} \right) | \tilde{\mathcal{G}}_1^{n,c} \right) ds = 0$$

where the last equality follows from the computation in (3). Whence $\mathbb{E}(R_{1,t}^{(n)}) = 0$, and $\text{Var}(R_{1,t}^{(n)}) = \mathbb{E}(\text{Var}(R_{1,t}^{(n)} | \tilde{\mathcal{G}}_1^{n,c}))$, so

$$\begin{aligned} \text{Var}(R_{1,t}^{(n)}) &= \mathbb{E} \left(\frac{1}{|J_n|} \sum_{j \in J_n} \text{Var} \left(\int_0^t G_{j,s} \left(\lambda_{j,s} - \hat{\lambda}_{j,s}^{(n)} \right) ds | \tilde{\mathcal{G}}_1^{n,c} \right) \right) \\ &= \mathbb{E} \left(\mathbb{E} \left(\left(\int_0^t G_s \left(\lambda_s - \hat{\lambda}_s^{(n)} \right) ds \right)^2 | \tilde{\mathcal{G}}_1^{n,c} \right) \right) \\ &= \mathbb{E} \left(\left(\int_0^t G_s \left(\lambda_s - \hat{\lambda}_s^{(n)} \right) ds \right)^2 \right) \\ &\leq (C')^2 \mathbb{E} \left(\int_0^t \left(\lambda_s - \hat{\lambda}_s^{(n)} \right)^2 ds \right) \\ &\leq (C')^2 h(n)^2 \end{aligned}$$

where we have used Assumption 4.1 (ii). Hence by Chebychev's inequality, it holds for all $\varepsilon > 0$ that

$$\mathbb{P}(|R_{1,t}^{(n)}| > \varepsilon) \leq \frac{\text{Var}(R_{1,t}^{(n)})}{\varepsilon^2} \leq \frac{(C')^2 h(n)^2}{\varepsilon^2} \rightarrow 0$$

as $n \rightarrow \infty$ uniformly over Θ by Assumption 4.2. This completes the first part of the proof. For the second part, we use a chaining argument based on the exponential inequality in Lemma A.6. We let

$$W_j^{s,t} = \frac{1}{t-s} \int_s^t G_{j,u} (\lambda_{j,u} - \hat{\lambda}_{j,u}^{(n)}) du$$

and

$$A = \frac{1}{\sqrt{|J_n|}} \sum_{j \in J_n} W_j^{s,t} = \frac{1}{t-s} (R_{1,t}^{(n)} - R_{1,s}^{(n)}).$$

Using that $(W_j^{s,t})_{j \in J_n}$ are i.i.d. conditionally on $\tilde{\mathcal{G}}_1^{n,c}$ we have by Lemma A.6 that $\mathbb{E}(A) = 0$ and that

$$\begin{aligned} \log \mathbb{E}(e^{xA}) &= \log \mathbb{E} \left(\mathbb{E} \left(e^{xA} \mid \tilde{\mathcal{G}}_1^{n,c} \right) \right) \\ &= \log \mathbb{E} \left(\prod_{j \in J_n} \mathbb{E} \left(e^{\frac{x}{\sqrt{|J_n|}} W_j^{s,t}} \mid \tilde{\mathcal{G}}_1^{n,c} \right) \right) \\ &\leq \log \mathbb{E} \left(e^{\frac{x^2 \nu}{2}} \right) \\ &= \frac{x^2 \nu}{2}. \end{aligned}$$

Hence A is also sub-Gaussian with variance factor ν . This implies that

$$\mathbb{P}(|A| > \eta) \leq 2e^{-\frac{\eta^2 \nu}{2}}$$

for all $\eta > 0$. Rephrased in terms of $R_1^{(n)}$ this bound reads

$$\mathbb{P} \left(|R_{1,t}^{(n)} - R_{1,s}^{(n)}| > \eta(t-s) \right) \leq 2e^{-\frac{\eta^2 \nu}{2}}$$

for all $\eta > 0$ and $s < t$. It now follows from the chaining lemma, Pollard (1984) Lemma VII.9, that $R_1^{(n)}$ is stochastic equicontinuous. Since the variance factor $\nu = (2CC')^2$ does not depend on $\theta \in \Theta$, we have stochastic equicontinuity uniformly over Θ by Corollary B.19. This completes the second part of the proof and we are done. \square

Note that the second part of the proof above establishes stochastic equicontinuity by a bound on the probability that the increments of the process are large. This is a well known technique, see, e.g., Example 2.2.12 in van der Vaart and Wellner (1996), from which the same conclusion will follow if

$$\mathbb{E}(|R_{1,t}^{(n)} - R_{1,s}^{(n)}|^p) \leq K|t-s|^{1+r}$$

for $K, p, r > 0$.

Proposition 4.4 now follows from combining the Lemmas A.7, A.4, and A.3. \square

A.6. *Proof of Proposition 4.5.* We separate the discussion of $D_1^{(n)}$ and $D_2^{(n)}$ into the Lemmas A.8 and A.10, respectively, which together amount to Proposition 4.5.

LEMMA A.8. *Suppose that Assumptions 4.1 and 4.2 hold. Then the stochastic process $\overline{D}^{(n)} := D_1^{(n)} - \sqrt{|J_n|} \cdot \gamma$ converges in distribution in $C[0, 1]$ uniformly over Θ .*

PROOF. Let $\overline{D}^{(n)} := D_1^{(n)} - \sqrt{|J_n|} \cdot \gamma$ and note that

$$\overline{D}^{(n)} = |J_n|^{-\frac{1}{2}} \sum_{j \in J_n} W_j,$$

where W_j is given by $W_{j,t} := \int_0^t G_{j,s}(\boldsymbol{\lambda}_{j,s} - \lambda_{j,s})ds - \gamma_t$ for each $j \in J_n$. By assumption, the variables $\{W_j : j \in J_n\}$ are i.i.d. with the same distribution as the process W given by $W_t := \int_0^t G_s(\boldsymbol{\lambda}_s - \lambda_s)ds - \gamma_t$. For each $\theta \in \Theta$, let Γ^θ be a Gaussian process with mean zero and covariance function $(s, t) \mapsto \text{cov}(W_s^\theta, W_t^\theta)$, which is well-defined by computations shown below.

We will show that $\overline{D}^{(n),\theta} \xrightarrow{\mathcal{D}/\Theta} \Gamma^\theta$ in $C[0, 1]$ by applying Lemma B.14, which is an example of Prokhorov's method of "tightness + identification of limit". We first prove that for any given $k \in \mathbb{N}$ and $0 \leq t_1 < t_2 < \dots < t_k \leq 1$,

$$\mathbf{D}^{(n)} := (\overline{D}_{t_1}^{(n)}, \overline{D}_{t_2}^{(n)}, \dots, \overline{D}_{t_k}^{(n)}) \xrightarrow{\mathcal{D}/\Theta} (\Gamma_{t_1}^\theta, \Gamma_{t_2}^\theta, \dots, \Gamma_{t_k}^\theta).$$

To this end we will apply the uniform CLT of [Lundborg, Shah and Peters \(2022, Proposition 19\)](#) to the sequence of random vectors $\mathbf{D}^{(n)} \in \mathbb{R}^k$, i.e., the sequence of normalized sums of i.i.d. copies of $\mathbf{W} := (W_{t_1}, \dots, W_{t_k})$. The process (W_t) is mean zero and hence \mathbf{W} is also mean zero. For any $t \in [0, 1]$ we observe that

$$\text{Var}(W_t) = \text{Var}(W_t + \gamma_t) \leq \mathbb{E} \left[\left(\int_0^t |G_s| \cdot |\boldsymbol{\lambda}_s - \lambda_s| ds \right)^2 \right] \leq 2C^2(C')^2.$$

Therefore the trace of $\text{Var}(\mathbf{W})$ is uniformly bounded, which implies the trace condition in Proposition 19 of [Lundborg, Shah and Peters \(2022\)](#). From Hölder's inequality and Minkowski's inequality, we note that for any $\mathbf{a}, \mathbf{b} \in \mathbb{R}^k$

$$\|\mathbf{a} + \mathbf{b}\|_2^3 \leq k^{3/2} \|\mathbf{a} + \mathbf{b}\|_3^3 \leq k^{3/2} (\|\mathbf{a}\|_3 + \|\mathbf{b}\|_3)^3 \leq 8k^{3/2} (\|\mathbf{a}\|_3^3 + \|\mathbf{b}\|_3^3).$$

Combining the above with Assumption 4.1 and Lemma A.2 yields that

$$\mathbb{E}[\|\mathbf{W}\|_2^3] \leq C_k \mathbb{E} \left[\left(\int_0^1 |G_s| \cdot |\boldsymbol{\lambda}_s - \lambda_s| ds \right)^3 \right] + C_k \sup_{t \in [0,1]} |\gamma_t|^3 \leq 16C_k C^3 (C')^3,$$

where $C_k = 8k^{5/2}$. Hence Proposition 19 of [Lundborg, Shah and Peters \(2022\)](#) lets us conclude that $\mathbf{D}^{(n)} \xrightarrow{\mathcal{D}/\Theta} \mathcal{N}(0, \text{Var}(\mathbf{W}))$. By definition of Γ^θ , this is equivalent to $\mathbf{D}^{(n)} \xrightarrow{\mathcal{D}/\Theta} (\Gamma_{t_1}^\theta, \Gamma_{t_2}^\theta, \dots, \Gamma_{t_k}^\theta)$.

We now argue that $(\overline{D}^{(n)})$ and (Γ^θ) are stochastically equicontinuous uniformly over Θ . From the definition of Γ^θ and by Assumption 4.1, it follows that

$$(4) \quad \mathbb{E}[(\Gamma_t^\theta - \Gamma_s^\theta)^2] = \mathbb{E}[(W_t - W_s)^2] \leq (2CC'(t-s))^2.$$

Hence $\frac{1}{t-s}(\Gamma_t^\theta - \Gamma_s^\theta)$ is Gaussian with a variance bounded over Θ and $0 \leq s < t \leq 1$. In particular, it is sub-Gaussian with a uniform variance factor over Θ and $0 \leq s < t \leq 1$. Since W is uniformly bounded over Θ , an application of Hoeffding's Lemma yields that $A_j^{t,s} :=$

$\frac{1}{t-s}(W_{j,t} - W_{j,s})$ is also sub-Gaussian with a variance factor ν that is uniform over Θ , $0 \leq s < t \leq 1$, and $j \in J_n$. Letting $A_{\bullet}^{s,t} = \frac{1}{t-s}(\overline{D}_t^{(n)} - \overline{D}_s^{(n)})$, we have

$$\mathbb{E}e^{xA_{\bullet}^{s,t}} = \prod_{j \in J_n} \mathbb{E}[e^{x|J_n|^{-1/2}A_j^{s,t}}] \leq \prod_{j \in J_n} e^{\frac{x^2\nu}{2|J_n|}} = e^{x^2\nu/2}.$$

Hence $A_{\bullet}^{s,t}$ is also sub-Gaussian with a variance factor uniformly over Θ and $0 \leq s < t \leq 1$.

From the uniform chaining lemma, Corollary B.19, we now conclude that both (Γ^θ) and $(\overline{D}^{(n)})$ are stochastically equicontinuous uniformly over Θ . By Proposition B.20, this means that the collection $(\overline{D}^{(n),\theta})$ is sequentially tight and that (Γ^θ) , which is constant in n , is uniformly tight.

Now we have shown convergence of the finite-dimensional marginals and appropriate tightness conditions, so Lemma B.14 lets us conclude that $\overline{D}^{(n)} \xrightarrow{\mathcal{D}/\Theta} \Gamma^\theta$ weakly in $C[0, 1]$. \square

Before moving on to the term $D_2^{(n)}$, we first note that Lemma A.8 implies that stochastic boundedness, as we will use this result in the proof of Theorem 4.6.

LEMMA A.9. *Suppose that Assumptions 4.1 and 4.2 hold. Then $\overline{D}^{(n)} := D_1^{(n)} - \sqrt{|J_n|} \cdot \gamma$ is stochastically bounded uniformly over Θ , i.e., for every $\varepsilon > 0$ there exists $K > 0$ such that*

$$\limsup_{n \rightarrow \infty} \sup_{\theta \in \Theta} \mathbb{P}\left(\|\overline{D}^{(n),\theta}\|_\infty > K\right) < \varepsilon.$$

PROOF. We have established in the proof of Lemma A.8, under the same conditions, that $\overline{D}^{(n),\theta} \xrightarrow{\mathcal{D}/\Theta} \Gamma^\theta$ weakly in $C[0, 1]$. By the uniform continuous mapping theorem formulated in Proposition B.6, it follows that $\|\overline{D}^{(n),\theta}\|_\infty \xrightarrow{\mathcal{D}/\Theta} \|\Gamma^\theta\|_\infty$. From Bengs and Holzmann (2019) Theorem 4.1 we then obtain that

$$\limsup_{n \rightarrow \infty} \sup_{\theta \in \Theta} \mathbb{P}(\|\overline{D}^{(n),\theta}\|_\infty > K) \leq \sup_{\theta \in \Theta} \mathbb{P}(\|\Gamma^\theta\|_\infty > K) \leq \frac{\mathbb{E}\|\Gamma^\theta\|_\infty}{K}.$$

Hence it suffices to argue that $\mathbb{E}\|\Gamma^\theta\|_\infty$ is uniformly bounded over Θ . To this end, we note that Equation (4) shows that square means of the increments of Γ^θ are smaller than those of a standard Brownian motion scaled by $2CC'$. Then the Sudakov–Fernique comparison inequality (Adler et al., 2007, Theorem 2.2.3) allows us to leverage this relationship to the expected uniform norms, i.e., $\mathbb{E}\|\Gamma^\theta\|_\infty \leq 2CC'\mathbb{E}(\sup_{t \in [0,1]} |B_t|)$. It can be verified that $\mathbb{E}(\sup_{t \in [0,1]} |B_t|)$ is finite, and in fact, equal to $\sqrt{\pi/2}$ as shown in saz (2019). \square

LEMMA A.10. *Suppose that Assumptions 4.1 and 4.2 hold, and that $G_t = X_t - \Pi_t$ is the additive residual process. Then $D_2^{(n)} \xrightarrow{\mathcal{P}/\Theta} 0$ in $D[0, 1]$ as $n \rightarrow \infty$.*

PROOF. Note first that the terms in $D_2^{(n)}$ are i.i.d. conditionally on $\tilde{\mathcal{G}}_1^{n,c}$, with the same distribution as the process ξ given by

$$\xi_t = \frac{1}{\sqrt{|J_n|}} \int_0^t (\hat{G}_s^{(n)} - G_s)(\lambda_s - \lambda_s) ds.$$

Since λ_t is independent of $\tilde{\mathcal{G}}_1^{n,c}$, we have from the innovation theorem that

$$\mathbb{E}(\lambda_t | \mathcal{F}_{t-} \vee \tilde{\mathcal{G}}_1^{n,c}) = \mathbb{E}(\lambda_t | \mathcal{F}_{t-}) = \lambda_t.$$

For the additive residual process we also note that $G_t - \hat{G}_t^{(n)} = \hat{\Pi}_t^{(n)} - \Pi_t$ is \mathcal{F}_t -predictable conditionally on $\tilde{\mathcal{G}}_1^{n,c}$. It now follows that

$$\begin{aligned} \sqrt{|J_n|} \cdot \mathbb{E}[\xi_t | \tilde{\mathcal{G}}_1^{n,c}] &= \int_0^t \mathbb{E}[(\hat{G}_s^{(n)} - G_s)(\lambda_s - \lambda_s) | \tilde{\mathcal{G}}_1^{n,c}] ds \\ &= \int_0^t \mathbb{E}[(\hat{G}_s^{(n)} - G_s)(\mathbb{E}[\lambda_s | \mathcal{F}_{s-} \vee \tilde{\mathcal{G}}_1^{n,c}] - \lambda_s) | \tilde{\mathcal{G}}_1^{n,c}] ds = 0. \end{aligned}$$

We can therefore conclude that $D_2^{(n)}$ is mean zero conditionally on $\tilde{\mathcal{G}}^{n,c}$. Using that the terms of $D_2^{(n)}$ are i.i.d. conditionally on $\tilde{\mathcal{G}}^{n,c}$ once more, we now obtain that

$$\begin{aligned} \text{Var}(D_{2,t}^{(n)} | \tilde{\mathcal{G}}^{n,c}) &= |J_n| \cdot \text{Var}(\xi_t | \tilde{\mathcal{G}}^{n,c}) = \mathbb{E} \left[\left(\int_0^t (\hat{G}_s^{(n)} - G_s)(\lambda_s - \lambda_s) ds \right)^2 \middle| \tilde{\mathcal{G}}^{n,c} \right] \\ &\leq 4C^2 \cdot \mathbb{E} \left(\int_0^1 (\hat{G}_s^{(n)} - G_s)^2 ds \middle| \tilde{\mathcal{G}}^{n,c} \right). \end{aligned}$$

Taking expectation of the above we have $\text{Var}(D_{2,t}^{(n)}) = \mathbb{E}(\text{Var}(D_{2,t}^{(n)} | \tilde{\mathcal{G}}^{n,c})) \leq 4C^2 g(n)^2$. By Chebyshev's inequality we get for all $\varepsilon > 0$

$$\mathbb{P} \left(|D_{2,t}^{(n)}| > \varepsilon \right) \leq \frac{4C^2 g(n)^2}{\varepsilon^2},$$

and by Assumption 4.2 we conclude that $D_{2,t}^{(n)} \xrightarrow{P/\Theta} 0$ for each $t \in [0, 1]$.

We now apply the same chaining argument used in the proofs of Lemma A.7 and Lemma A.8. From Assumption 4.1, we have for $0 \leq s < t \leq 1$ that $|\xi_t - \xi_s| \leq 4\sqrt{|J_n|}CC'(t-s)$. Hence the conditional Hoeffding's lemma (Lemma A.5) yields that

$$A_j^{s,t} = \frac{1}{t-s} \int_s^t (\hat{G}_{j,s}^{(n)} - G_{j,s})(\lambda_{j,s} - \lambda_{j,s}) ds$$

is sub-Gaussian conditionally on $\tilde{\mathcal{G}}_1^{n,c}$ with a variance factor ν that is uniform over Θ and $s < t$ (cf. the proof of Lemma A.6). Letting $A_{\bullet}^{s,t} = \frac{1}{t-s} (D_{2,t}^{(n)} - D_{2,s}^{(n)})$, we have for any $x \in \mathbb{R}$

$$\begin{aligned} \mathbb{E} \left(e^{xA_{\bullet}^{s,t}} \right) &= \mathbb{E} \left(\mathbb{E} \left[e^{xA_{\bullet}^{s,t}} \middle| \tilde{\mathcal{G}}_1^{n,c} \right] \right) \\ &= \mathbb{E} \left(\prod_{j \in J_n} \mathbb{E} \left[e^{x|J_n|^{-1/2} A_j^{s,t}} \middle| \tilde{\mathcal{G}}_1^{n,c} \right] \right) \leq \prod_{j \in J_n} e^{\frac{x^2 \nu}{2|J_n|}} = e^{x^2 \nu / 2}, \end{aligned}$$

so $A_{\bullet}^{s,t}$ is also sub-Gaussian uniformly over $s < t$ and Θ . In terms of $D_2^{(n)}$, this means that we can apply the uniform chaining lemma, Corollary B.19, and conclude that it is stochastically equicontinuous uniformly over Θ .

Since $D_{2,t}^{(n)} \xrightarrow{P/\Theta} 0$ for each $t \in [0, 1]$ and $(D_2^{(n)})$ is stochastically equicontinuous uniformly over Θ , Lemma B.16 now lets us conclude that $\sup_{t \in [0,1]} |D_{2,t}^{(n)}| \xrightarrow{P/\Theta} 0$ and we are done. \square

A.7. Proof of Theorem 4.6. Before proving Theorem 4.6, we first prove that the collection of Gaussian martingales from Proposition 4.3 is tight in $C[0, 1]$ (see Definition B.7).

LEMMA A.11. *Let $(U^\theta)_{\theta \in \Theta}$ be the collection of Gaussian martingales from Proposition 4.3, i.e., U^θ is a mean zero continuous Gaussian martingale with variance function \mathcal{V}^θ . Under Assumption 4.1, $(U^\theta)_{\theta \in \Theta}$ is uniformly tight in $C[0, 1]$.*

PROOF. We will use Theorem 7.3 in Billingsley (2013), which characterizes tightness of measures in $C[0, 1]$. The first condition of the theorem is trivially satisfied for $(U^\theta)_{\theta \in \Theta}$ since $\mathbb{P}(U_0^\theta = 0) = 1$ for all $\theta \in \Theta$.

By Proposition C.2, U^θ has a distributional representation as a time-transformed Brownian motion such that $(U_t^\theta)_{t \in [0, 1]} \stackrel{\mathcal{D}}{=} (B_{\mathcal{V}^\theta(t)})_{t \in [0, 1]}$, where B is a Brownian motion. Recall that Brownian motion is α -Hölder continuous for $\alpha \in (0, \frac{1}{2})$, which means that

$$K(\alpha) = \sup_{s \neq t} \frac{|B_t - B_s|}{|t - s|^\alpha} < \infty.$$

Note also that the collection of variance functions is uniformly Lipschitz by Lemma A.2 with uniform Lipschitz constant C_0 , say. It follows that for every $\varepsilon > 0$,

$$\begin{aligned} \lim_{\delta \rightarrow 0^+} \sup_{\theta \in \Theta} \mathbb{P} \left(\sup_{|t-s| < \delta} |U_t^\theta - U_s^\theta| > \varepsilon \right) &= \lim_{\delta \rightarrow 0^+} \sup_{\theta \in \Theta} \mathbb{P} \left(\sup_{|t-s| < \delta} |B_{\mathcal{V}^\theta(t)} - B_{\mathcal{V}^\theta(s)}| > \varepsilon \right) \\ &\leq \lim_{\delta \rightarrow 0^+} \sup_{\theta \in \Theta} \mathbb{P} \left(K(\alpha) \sup_{|t-s| < \delta} |\mathcal{V}^\theta(t) - \mathcal{V}^\theta(s)|^\alpha > \varepsilon \right) \\ &= \lim_{\delta \rightarrow 0^+} \mathbb{P} \left(K(\alpha) C_0^\alpha \delta^\alpha > \varepsilon \right) = 0. \end{aligned}$$

This establishes the second condition of Theorem 7.3 in Billingsley (2013), and we thus conclude that $(U^\theta)_{\theta \in \Theta}$ is uniformly tight in $C[0, 1]$. \square

We now return to the proof of Theorem 4.6.

For part i), we first note that under H_0 we can take $\lambda_t = \lambda_t$, which implies that both $D_1^{(n)}$ and $D_2^{(n)}$ equal the zero-process.

Combining Propositions 4.3 and 4.4 with the uniform version of Slutsky's theorem formulated in Lemma B.5, we conclude that

$$\sqrt{|J_n|} \hat{\gamma}^{(n)} = \underbrace{U^{(n)}}_{\xrightarrow{\mathcal{D}/\Theta_0} U^\theta} + \underbrace{R_1^{(n)} + R_2^{(n)} + R_3^{(n)}}_{\xrightarrow{\mathcal{P}/\Theta} 0} + \underbrace{D_1^{(n)} + D_2^{(n)}}_{=0 \text{ under } H_0} \xrightarrow{\mathcal{P}/\Theta_0} U^\theta,$$

in $D[0, 1]$ as $n \rightarrow \infty$, where U^θ is the Gaussian martingale from Proposition 4.3.

For part ii) we can, in addition to Propositions 4.3 and 4.4, apply Proposition 4.5 and Lemma A.9. Using the triangle inequality on the decomposition (17) yields that

$$\begin{aligned} \sqrt{|J_n|} \cdot \|\hat{\gamma}^{(n)} - \gamma\|_\infty &\leq \|U^{(n)}\|_\infty + \|D_1^{(n)} - \sqrt{|J_n|} \gamma\|_\infty \\ &\quad + \|R_1^{(n)}\|_\infty + \|R_2^{(n)}\|_\infty + \|R_3^{(n)}\|_\infty + \|D_2^{(n)}\|_\infty. \end{aligned}$$

All the terms in the second line converge in probability to zero uniformly over Θ . Combined with the convergences established in Proposition 4.3 and Lemma A.8, we obtain that

$$\begin{aligned} &\limsup_{n \rightarrow \infty} \sup_{\theta \in \Theta} \mathbb{P} \left(\sqrt{|J_n|} \cdot \|\hat{\gamma}^{(n), \theta} - \gamma^\theta\|_\infty > K \right) \\ (5) \quad &\leq \sup_{\theta \in \Theta} \mathbb{P} \left(\|U^\theta\|_\infty > K/6 \right) + \sup_{\theta \in \Theta} \mathbb{P} \left(\|\Gamma^\theta\|_\infty > K/6 \right), \end{aligned}$$

where Γ^θ is the limiting Gaussian process from (the proof of) Lemma A.8. The last term in (5) can be made arbitrarily small for K sufficiently large by Lemma A.9. Lemma A.11 states that the family $(U^\theta)_{\theta \in \Theta}$ is tight in $C[0, 1]$, and hence the family $(\|U^\theta\|_\infty)_{\theta \in \Theta}$ is tight in $\mathbb{R}_{\geq 0}$. This implies that the first term in (5) can also be made arbitrarily small for K sufficiently large. This establishes (26) and we are done. \square

A.8. *Proof of Proposition 4.7.* Consider the decomposition of the variance function estimator given by

$$\hat{\mathcal{V}}_n(t) = A_t^{(n)} + B_t^{(n)} + 2C_t^{(n)}$$

where

$$\begin{aligned} A_t^{(n)} &= \frac{1}{|J_n|} \sum_{j \in J_n} \int_0^t G_{j,s}^2 dN_{j,s}, \\ B_t^{(n)} &= \frac{1}{|J_n|} \sum_{j \in J_n} \int_0^t \left(G_{j,s} - \hat{G}_{j,s}^{(n)} \right)^2 dN_{j,s}, \\ C_t^{(n)} &= \frac{1}{|J_n|} \sum_{j \in J_n} \int_0^t G_{j,s} \left(G_{j,s} - \hat{G}_{j,s}^{(n)} \right) dN_{j,s}. \end{aligned}$$

We first consider the asymptotic limit of $A^{(n)}$, which is the empirical mean of $|J_n|$ i.i.d. samples of the process $\int_0^t G_s^2 dN_s$. Under Assumption 4.1, we can apply the first part of Lemma A.1 which states $\mathbf{M}_t^2 - \mathbf{\Lambda}_t$ is a martingale. We use this fact to note that

$$\mathbb{E}(N_1^2) = \mathbb{E}((\mathbf{M}_1 + \mathbf{\Lambda}_1)^2) \leq 2(\mathbb{E}(\mathbf{M}_1^2) + \mathbb{E}(\mathbf{\Lambda}_1^2)) = 4\mathbb{E}\left(\left(\int_0^1 \lambda_s ds\right)^2\right) \leq 4C^2.$$

Now, another use of Assumption 4.1 shows that $\int_0^t G_s^2 dN_s$ has a second moment bounded by $4(CC')^2$. Thus we can apply the uniform law of large numbers (Shah and Peters, 2020, Lemma 19) to conclude for each $t \in [0, 1]$,

$$A_t^{(n)} = \frac{1}{|J_n|} \sum_{j \in J_n} \int_0^t G_{j,s}^2 dN_{j,s} \xrightarrow{P/\Theta} \mathbb{E}\left(\int_0^t G_s^2 dN_s\right) = \mathcal{V}(t).$$

Note also that $A^{(n)}$ and \mathcal{V} are non-decreasing and that the collection $(\mathcal{V}^\theta)_{\theta \in \Theta}$ is uniformly equicontinuous by Lemma A.2. These are exactly the conditions for Lemma B.13, so we can automatically conclude that $\sup_{t \in [0, 1]} |A_t^{(n)} - \mathcal{V}(t)| \xrightarrow{P/\Theta} 0$.

Next we show that the remainder terms $B^{(n)}$ and $C^{(n)}$ converge uniformly to zero in expectation. Similarly to the proof of Lemma A.4, we have under Assumptions 4.1 and 4.2,

$$\begin{aligned} \mathbb{E}\left(\sup_{0 \leq t \leq 1} B_t^{(n)}\right) &= \mathbb{E}(B_1^{(n)}) = \mathbb{E}\left(\frac{1}{|J_n|} \sum_{j \in J_n} \int_0^1 \left(G_{j,s} - \hat{G}_{j,s}^{(n)}\right)^2 \lambda_{j,s} ds\right) \\ &= \mathbb{E}\left(\mathbb{E}\left(\frac{1}{|J_n|} \sum_{j \in J_n} \int_0^1 \left(G_{j,s} - \hat{G}_{j,s}^{(n)}\right)^2 \lambda_{j,s} ds \mid \tilde{\mathcal{G}}_1^c\right)\right) \\ &= \mathbb{E}\left(\int_0^1 \left(G_s - \hat{G}_s^{(n)}\right)^2 \lambda_s ds\right) \\ &\leq C \cdot g(n)^2 \longrightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$ uniformly over Θ . Lastly, we see that

$$\mathbb{E}\left|\sup_{0 \leq t \leq 1} C_t^{(n)}\right| \leq \mathbb{E}\left(\sup_{0 \leq t \leq 1} |C_t^{(n)}|\right)$$

$$\begin{aligned}
&\leq \mathbb{E} \left(\frac{1}{|J_n|} \sum_{j \in J_n} \sup_{0 \leq t \leq 1} \int_0^t |G_{j,s}| |G_{j,s} - \hat{G}_{j,s}^{(n)}| \boldsymbol{\lambda}_{j,s} ds \right) \\
&= \mathbb{E} \left(\frac{1}{|J_n|} \sum_{j \in J_n} \int_0^1 |G_{j,s}| |G_{j,s} - \hat{G}_{j,s}^{(n)}| \boldsymbol{\lambda}_{j,s} ds \right) \\
&= \mathbb{E} \left(\mathbb{E} \left(\frac{1}{|J_n|} \sum_{j \in J_n} \int_0^1 |G_{j,s}| |G_{j,s} - \hat{G}_{j,s}^{(n)}| \boldsymbol{\lambda}_{j,s} ds \mid \tilde{\mathcal{G}}_1^c \right) \right) \\
&= \mathbb{E} \left(\int_0^1 |G_s| |G_s - \hat{G}_s^{(n)}| \boldsymbol{\lambda}_s ds \right) \\
&\leq CC' \mathbb{E} \left(\int_0^1 |G_s - \hat{G}_s^{(n)}| ds \right) \\
&\leq CC' \cdot g(n) \rightarrow 0
\end{aligned}$$

as $n \rightarrow \infty$ uniformly over Θ by Assumption 4.2. Combining the convergences established for $A^{(n)}$, $B^{(n)}$, and $C^{(n)}$, we get by a generalized Slutsky (Lemma B.11) that

$$\sup_{t \in [0,1]} |\hat{\mathcal{V}}_n(t) - \mathcal{V}(t)| \xrightarrow{P/\Theta} 0.$$

□

A.9. *Proof of Corollary 4.8.* Under Assumptions 4.1 and 4.2 we know by Theorem 4.6 and Proposition 4.7 that

$$(6) \quad \sqrt{|J_n|} \hat{\gamma}^{(n),\theta} \xrightarrow{\mathcal{D}/\Theta_0} U^\theta \quad \text{and} \quad \hat{\mathcal{V}}_n^\theta \xrightarrow{P/\Theta_0} \mathcal{V}^\theta$$

in $D[0,1]$ as $n \rightarrow \infty$. If we were to show pointwise convergence of the test statistic, this would now be a straightforward consequence of the continuous mapping theorem. However, to show uniform convergence, we will need an additional tightness argument.

Let $(\theta_n)_{n \in \mathbb{N}} \subset \Theta_0$ be an arbitrary sequence. Proposition B.3 then states that it suffices to show that there exists a subsequence $(\theta_{k(n)})_{n \in \mathbb{N}} \subseteq (\theta_n)_{n \in \mathbb{N}}$, with $k: \mathbb{N} \rightarrow \mathbb{N}$ strictly increasing, such that

$$(7) \quad \lim_{n \rightarrow \infty} d_{BL}(\hat{D}_{k(n)}^{\theta_{k(n)}}, \mathcal{J}(U^{\theta_{k(n)}}, \mathcal{V}^{\theta_{k(n)}})) = 0.$$

Here d_{BL} denotes the bounded Lipschitz metric defined in Section B. By Lemma A.11, the collection $(U^\theta)_{\theta \in \Theta}$ is tight in $C[0,1]$ under Assumption 4.1. Therefore, Prokhorov's theorem (Kallenberg, 2021, Theorem 23.2) asserts that there exists a subsequence $(\theta_{a(n)}) \subset (\theta_n)$, and a $C[0,1]$ -valued random variable \tilde{U} such that $U^{\theta_{a(n)}} \xrightarrow{\mathcal{D}} \tilde{U}$ in $C[0,1]$.

Likewise, Lemma A.2 states that the collection $(V^\theta)_{\theta \in \Theta}$ is uniformly bounded and uniformly equicontinuous under Assumption 4.1. Thus the Arzelà-Ascoli theorem yields that there exists a further subsequence $(\theta_{b(n)}) \subset (\theta_{a(n)})$ and a function $\tilde{\mathcal{V}} \in C[0,1]$ such that $\|\mathcal{V}^{\theta_{b(n)}} - \tilde{\mathcal{V}}\|_\infty \rightarrow 0$.

Combining the convergences of $U^{\theta_{b(n)}}$ and $\mathcal{V}^{\theta_{b(n)}}$ with those in Equation (6), it follows from the triangle inequality of the metric d_{BL} that also

$$\sqrt{|J_{b(n)}|} \hat{\gamma}^{(b(n)),\theta_{b(n)}} \xrightarrow{\mathcal{D}} \tilde{U} \quad \text{and} \quad \hat{\mathcal{V}}_{b(n)}^{\theta_{b(n)}} \xrightarrow{P} \tilde{\mathcal{V}},$$

in $D[0, 1]$ as $n \rightarrow \infty$. Now we may use that convergence in Skorokhod topology is equivalent to convergence in uniform topology whenever the limit variable continuous, see e.g. [Kallenberg \(2021, Theorem 23.9\)](#). Hence the convergences above also hold in $(D[0, 1], \|\cdot\|_\infty)$.

Since \mathcal{V} is deterministic, this implies the joint convergences

$$(U^{\theta_{b(n)}}, \mathcal{V}^{\theta_{b(n)}}) \xrightarrow{\mathcal{D}} (\tilde{U}, \tilde{\mathcal{V}}) \quad \text{and} \quad \left(\sqrt{|J_{b(n)}|} |\hat{\gamma}^{(b(n)), \theta_{b(n)}}|, \hat{\mathcal{V}}_{b(n)}^{\theta_{b(n)}} \right) \xrightarrow{\mathcal{D}} (\tilde{U}, \tilde{\mathcal{V}})$$

in the product space $D[0, 1] \times D[0, 1]$ endowed with the uniform topology. Since $(\tilde{U}, \tilde{\mathcal{V}}) \in C[0, 1] \times \{\mathcal{V}^\theta : \theta \in \Theta_0\}$ takes values in the continuity set of \mathcal{J} by assumption, the classical continuous mapping theorem lets us conclude that

$$\mathcal{J}(U^{\theta_{b(n)}}, \mathcal{V}^{\theta_{b(n)}}) \xrightarrow{\mathcal{D}} \mathcal{J}(\tilde{U}, \tilde{\mathcal{V}}) \quad \text{and} \quad \hat{D}_{b(n)}^{\theta_{b(n)}} = \mathcal{J}\left(\sqrt{|J_{b(n)}|} |\hat{\gamma}^{(b(n)), \theta_{b(n)}}|, \hat{\mathcal{V}}_{b(n)}^{\theta_{b(n)}}\right) \xrightarrow{\mathcal{D}} \mathcal{J}(\tilde{U}, \tilde{\mathcal{V}})$$

as $n \rightarrow \infty$. Now another application of the triangle inequality with $\mathcal{J}(\tilde{U}, \tilde{\mathcal{V}})$ as intermediate value shows that (7) holds with $k(n) = b(n)$, so we are done. \square

A.10. Proof of Theorem 5.1. We will apply Corollary 4.8 with the functional \mathcal{J} given by

$$\mathcal{J}(f_1, f_2) = \mathbb{1}(f_2 \neq 0) \frac{\|f_1\|_\infty}{\sqrt{|f_2(1)|}}, \quad f_1, f_2 \in D[0, 1].$$

Under Assumption 5.1, it suffices to check continuity of \mathcal{J} on the set Υ given by

$$\Upsilon := C[0, 1] \times \{f \in C[0, 1] \mid \delta_1 \leq |f(1)|\} \supset C[0, 1] \times \overline{\{\mathcal{V}^\theta : \theta \in \Theta_0\}}.$$

To see that \mathcal{J} is continuous on Υ in the uniform topology, we note that it can be written as a composition of the continuous maps

$$\begin{aligned} \Upsilon &\longrightarrow [0, \infty) \times [\delta_1, \infty), & (f_1, f_2) &\mapsto (\|f_1\|_\infty, |f_2(1)|), \\ [0, \infty) \times [\delta_1, \infty) &\longrightarrow \mathbb{R}, & (x_1, x_2) &\mapsto \frac{x_1}{\sqrt{x_2}}. \end{aligned}$$

Thus it follows from Corollary 4.8 that

$$\hat{T}_n = \frac{\sqrt{|J_n|} \sup_{t \in [0, 1]} |\hat{\gamma}_t^{(n)}|}{\sqrt{\hat{\mathcal{V}}_n(1)}} = \mathcal{J}\left(\sqrt{|J_n|} |\hat{\gamma}^{(n)}|, \hat{\mathcal{V}}_n\right) \xrightarrow{\mathcal{D}/\Theta_0} \mathcal{J}(U, \mathcal{V}) = \frac{\|U\|_\infty}{\mathcal{V}(1)}.$$

With (B_u) a Brownian motion it follows by Proposition C.2 that

$$(8) \quad \frac{\|U\|_\infty}{\sqrt{\mathcal{V}(1)}} \stackrel{\mathcal{D}}{=} \frac{\sup_{0 \leq t \leq 1} |B_{\mathcal{V}(t)}|}{\sqrt{\mathcal{V}(1)}} = \frac{\sup_{0 \leq u \leq \mathcal{V}(1)} |B_u|}{\sqrt{\mathcal{V}(1)}} \stackrel{\mathcal{D}}{=} \sup_{0 \leq t \leq 1} |B_t| \stackrel{\mathcal{D}}{=} S,$$

where we have used that \mathcal{V} is continuous and that Brownian motion is scale invariant. This establishes the first part of the theorem.

For the second part, we first note that the distribution of S is absolutely continuous with respect to Lebesgue measure, which follows from Equation (31). Then we can use Theorem 4.1 of [Bengs and Holzmann \(2019\)](#) to conclude that

$$\limsup_{n \rightarrow \infty} \sup_{\theta \in \Theta} |\mathbb{P}(\hat{T}_n \leq z_{1-\alpha}) - (1 - \alpha)| = 0.$$

It follows from the triangle inequality that

$$\limsup_{n \rightarrow \infty} \sup_{\theta \in \Theta} \mathbb{P}(\Psi_n^\alpha = 1) = \limsup_{n \rightarrow \infty} \sup_{\theta \in \Theta} \mathbb{P}(\hat{T}_n > z_{1-\alpha}) \leq \alpha.$$

\square

A.11. *Proof of Theorem 5.2.* Let $0 < \alpha < \beta < 1$ be given. The second part of Theorem 4.6 permits us to choose $K > 0$ sufficiently large such that

$$(9) \quad \limsup_{n \rightarrow \infty} \sup_{\theta \in \Theta} \mathbb{P} \left((\sqrt{|J_n|} \|\hat{\gamma}^{(n),\theta} - \gamma^\theta\|_\infty) > K \right) < 1 - \beta.$$

We then choose $c > K + z_{1-\alpha} \sqrt{1 + C(C')^2}$ such that for all $\theta \in \mathcal{A}_{c,n}$, it holds that

$$\sqrt{|J_n|} \|\gamma^\theta\|_\infty - z_{1-\alpha} \sqrt{1 + \mathcal{V}^\theta(1)} \geq c - z_{1-\alpha} \sqrt{1 + C(C')^2} > K,$$

where we have used Lemma A.2 in the first inequality. The (reverse) triangle inequality now yields that for any $\theta \in \mathcal{A}_{c,n}$

$$\begin{aligned} (\Psi_n^\theta = 0) &= (\hat{T}_n^\theta \leq z_{1-\alpha}) = \left(\|\hat{\gamma}^{(n),\theta}\|_\infty \leq \sqrt{\hat{\mathcal{V}}_n^\theta(1)} \frac{z_{1-\alpha}}{\sqrt{|J_n|}} \right) \\ &\subseteq \left(\|\gamma^\theta\|_\infty - \|\hat{\gamma}^{(n),\theta} - \gamma^\theta\|_\infty \leq \sqrt{\hat{\mathcal{V}}_n^\theta(1)} \frac{z_{1-\alpha}}{\sqrt{|J_n|}} \right) \\ &\subseteq E_1^{(n),\theta} \cup E_2^{(n),\theta}, \end{aligned}$$

where

$$\begin{aligned} E_1^{(n),\theta} &= \left(\sqrt{|J_n|} \|\hat{\gamma}^{(n),\theta} - \gamma^\theta\|_\infty > K \right), \\ E_2^{(n),\theta} &= \left(\hat{\mathcal{V}}_n^\theta(1) > 1 + \mathcal{V}^\theta(1) \right) \subseteq \left(|\hat{\mathcal{V}}_n^\theta(1) - \mathcal{V}^\theta(1)| > 1 \right). \end{aligned}$$

From Proposition 4.7 we know that $\limsup_{n \rightarrow \infty} \sup_{\theta \in \Theta} \mathbb{P}(E_2^{(n),\theta}) = 0$, so from the choice of K we conclude that

$$\limsup_{n \rightarrow \infty} \sup_{\theta \in \Theta} \mathbb{P}(\Psi_n = 0) \leq \limsup_{n \rightarrow \infty} \sup_{\theta \in \Theta} \mathbb{P}(E_1^{(n),\theta}) < 1 - \beta.$$

The desired statement follows from substituting $\mathbb{P}(\Psi_n = 0) = 1 - \mathbb{P}(\Psi_n = 1)$ into the above equation and simplifying. \square

A.12. *Proof of Theorem 5.4.* Assume that H_0 holds and note that Assumptions 4.1 and 4.2 are satisfied for every sample split $J_n^k \cup (J_n^k)^c$, $k = 1, \dots, K$.

We consider the decomposition in Equation (17) for each sample split $J_n^k \cup (J_n^k)^c$, and denote the corresponding processes by $U^{k,(n)}$, $R_1^{k,(n)}$, $R_2^{k,(n)}$, $R_3^{k,(n)}$, $D_1^{k,(n)}$, and $D_2^{k,(n)}$. For each fold $k \in \{1, \dots, K\}$, we can then apply the results in Section 4 for a single data split:

- By Proposition 4.3, we have that $U^{k,(n)} \xrightarrow{\mathcal{D}/\Theta} U$ in $D[0, 1]$, where U is a mean zero continuous Gaussian martingale with variance function \mathcal{V} .
- By Proposition 4.4, $R_\ell^{k,(n)} \xrightarrow{P/\Theta} 0$ in $(D[0, 1], \|\cdot\|_\infty)$ as $n \rightarrow \infty$.
- Under H_0 , the processes $D_1^{k,(n)}$ and $D_2^{k,(n)}$ are equal to the zero process almost surely.

Recall that the folds are assumed to have uniform asymptotic density, which is equivalent to $\frac{\sqrt{n}}{\sqrt{K|J_n^k|}} \rightarrow 1$ as $n \rightarrow \infty$. Thus we may also conclude that for each fixed k and ℓ ,

$$\frac{\sqrt{n}}{\sqrt{K|J_n^k|}} U^{k,(n)} \xrightarrow{\mathcal{D}/\Theta} U \quad \text{and} \quad \frac{\sqrt{n}}{K\sqrt{|J_n^k|}} R_\ell^{k,(n)} \xrightarrow{P/\Theta} 0,$$

where the convergences hold in the Skorokhod and uniform topology, respectively. Now the key observation is that

$$U^{1,(n)} \perp\!\!\!\perp \dots \perp\!\!\!\perp U^{K,(n)}.$$

To see this, note that $U^{k,(n)}$ is constructed from $(G_j, M_j)_{j \in J^k}$ only, and by the i.i.d. assumption of the data, the collections $(G_j, M_j)_{j \in J^1}, \dots, (G_j, M_j)_{j \in J^K}$ are jointly independent. We can therefore apply Lemma B.12 iteratively to the sequences

$$\frac{\sqrt{n}}{\sqrt{K|J_n^1|}} U^{1,(n)}, \dots, \frac{\sqrt{n}}{\sqrt{K|J_n^K|}} U^{K,(n)}$$

to conclude that their sum is uniformly convergent to the sum of K independent copies of U . Using the convolution property of the Gaussian distribution, it therefore follows that

$$\check{U}^{K,(n)} := \frac{1}{\sqrt{K}} \sum_{k=1}^K \frac{\sqrt{n}}{\sqrt{K|J_n^k|}} U^{k,(n)} \xrightarrow{\mathcal{D}/\Theta} U$$

in $D[0, 1]$ as $n \rightarrow \infty$. By the uniform Slutsky theorem formulated in Lemma B.11, we can therefore conclude that

$$\sqrt{n}\check{\gamma}^{K,(n)} = \check{U}^{K,(n)} + \sum_{k=1}^K \frac{\sqrt{n}}{K\sqrt{|J_n^k|}} \left(R_1^{k,(n)} + R_2^{k,(n)} + R_3^{k,(n)} + D_1^{k,(n)} + D_2^{k,(n)} \right) \xrightarrow{\mathcal{D}/\Theta_0} U$$

in $D[0, 1]$ as $n \rightarrow \infty$. Note that we use $\theta \in \Theta_0$ to ensure that $D_1^{k,(n)} + D_2^{k,(n)}$ is equal to the zero process almost surely. Since the limit $(U^\theta)_{\theta \in \Theta_0}$ is tight in $C[0, 1]$ by Lemma A.11, Proposition B.9 lets us conclude that $\sqrt{n}\|\check{\gamma}^{K,(n)}\|_\infty \xrightarrow{\mathcal{D}/\Theta_0} \|U\|_\infty$.

Consider now the cross-fitted variance estimator at its endpoint

$$\check{V}_{K,n}(1) = \frac{1}{K} \sum_{k=1}^K \frac{1}{|J_n^k|} \sum_{j \in J_n^k} \int_0^1 \left(\hat{G}_{j,s}^{k,(n)} \right)^2 dN_{j,s}.$$

From Proposition 4.7, we see that $\check{V}_{K,n}(1)$ is an average of K variables converging uniformly in probability to $\mathcal{V}(1)$ in the uniform topology. Hence $\check{V}_{K,n}(1)$ also converges uniformly in probability to $\mathcal{V}(1)$ in the uniform topology. We can then apply Theorem 6.3 of Bengs and Holzmann (2019), which is a uniform version of Slutsky's theorem, to conclude that

$$\check{T}_n^K = \frac{\sqrt{n}\|\check{\gamma}^{K,(n)}\|_\infty}{\check{V}_{K,n}(1)} \xrightarrow{\mathcal{D}/\Theta_0} \frac{\|U\|_\infty}{\mathcal{V}(1)} \stackrel{\mathcal{D}}{=} S,$$

as $n \rightarrow \infty$, where last equality in distribution was established in (8).

Following the second part of the proof of Theorem 5.1, we conclude the X-LCT has uniform asymptotic level. \square

B. Uniform stochastic convergence. In this section, we discuss weak convergence of random variables with values in a metric space uniformly over a parameter set Θ . The uniformity over the parameter set can be used, for example, to establish uniform asymptotic level as well as power under local alternatives.

The content of this section extends the works of Bengs and Holzmann (2019) and Kasy (2019), and we especially build upon Appendix B of Lundborg, Shah and Peters (2022), in which uniform stochastic convergence is considered in *separable* Banach spaces and Hilbert spaces. The space $(D[0, 1], \|\cdot\|_\infty)$ of càdlàg functions endowed with the uniform norm is a Banach space, but it is unfortunately not separable. Therefore we extend the notion of

uniform stochastic convergence to random variables in metric spaces, with the condition that the limit is supported on a separable set. This allows to consider uniform weak convergence in two important special cases: i) convergence in $(D[0, 1], \|\cdot\|_\infty)$ towards variables in $(C[0, 1], \|\cdot\|_\infty)$, and ii) convergence in $D[0, 1]$ endowed with the Skorokhod metric.

The Skorokhod space $D[0, 1]$ is, if not specified otherwise, equipped with the complete Skorokhod metric d° , which makes it a Polish space, i.e., a complete and separable metric space. See for example Section 12 in [Billingsley \(2013\)](#) for a discussion of the Skorokhod space and in particular Equation (12.16) for a definition of d° .

B.1. Uniform stochastic convergence in metric spaces. Throughout this section we consider a background probability space $(\Omega, \mathbb{F}, \mathbb{P})$ and let $(\mathbb{D}, d_{\mathbb{D}})$ denote a generic metric space. We define $BL_1(\mathbb{D})$ as the set of real-valued functions on \mathbb{D} with Lipschitz norm bounded by 1, that is, functions $f: \mathbb{D} \rightarrow \mathbb{R}$ with $\|f\|_\infty \leq 1$ and $|f(x) - f(y)| \leq d_{\mathbb{D}}(x, y)$ for every $x, y \in \mathbb{D}$. Let $\mathcal{M}_1(\mathbb{D})$ denote the set of Borel probability measures on \mathbb{D} . We then define the *bounded Lipschitz metric* on $\mathcal{M}_1(\mathbb{D})$ by

$$d_{BL}(\mu, \nu) := \sup_{f \in BL_1(\mathbb{D})} \left| \int f d\mu - \int f d\nu \right|, \quad \mu, \nu \in \mathcal{M}_1(\mathbb{D}).$$

For any pair (X, Y) of \mathbb{D} -valued random variables we use the shorthand notation

$$d_{BL}(X, Y) = d_{BL}(X(\mathbb{P}), Y(\mathbb{P})) = \sup_{f \in BL_1(\mathbb{D})} |\mathbb{E}(f(X) - f(Y))|.$$

If the underlying metric space is ambiguous for d_{BL} , we will specify that it is the bounded Lipschitz metric on $\mathcal{M}_1(\mathbb{D})$ by writing $d_{BL(\mathbb{D})}$. Our interest in the bounded Lipschitz metric is due to its characterization of weak convergence.

PROPOSITION B.1. *Let X, X_1, X_2, \dots be a sequence of \mathbb{D} -valued random variables. Assume that there exists a separable subset $\mathbb{D}_0 \subseteq \mathbb{D}$ such that $\mathbb{P}(X \in \mathbb{D}_0) = 1$. Then the following are equivalent:*

- *The sequence $(X_n)_{n \geq 1}$ converges in distribution to X , i.e., for all $f \in C_b(\mathbb{D})$ it holds that $\mathbb{E}[f(X_n)] \rightarrow \mathbb{E}[f(X)]$ as $n \rightarrow \infty$.*
- *It holds that $d_{BL}(X_n, X) \rightarrow 0$ as $n \rightarrow \infty$.*

PROOF. See Theorem 1.12.2, Addendum 1.12.3, and the following discussion in [van der Vaart and Wellner \(1996\)](#). □

To discuss uniform stochastic convergence, we will for the remaining part of this section let Θ be fixed set, which is used as a (possible) parameter set for every random variable. We say that a collection $(X^\theta)_{\theta \in \Theta}$ of \mathbb{D} -valued random variables is *separable* if there exists a separable subset $\mathbb{D}_0 \subseteq \mathbb{D}$ such that $\mathbb{P}(X^\theta \in \mathbb{D}_0) = 1$ for all $\theta \in \Theta$. If \mathbb{D} is a separable metric space, then any collection of \mathbb{D} -valued random variables is automatically separable.

Now Lemma [B.1](#) justifies the following generalization of weak convergence uniformly over Θ :

DEFINITION B.2. Let $(X_n^\theta)_{n \in \mathbb{N}, \theta \in \Theta}$ and $(X^\theta)_{\theta \in \Theta}$ be collections of \mathbb{D} -valued random variables and assume that $(X^\theta)_{\theta \in \Theta}$ is separable. We say that:

- (i) X_n^θ converges uniformly in distribution over Θ to X^θ in \mathbb{D} , and write $X_n^\theta \xrightarrow{D/\Theta} X^\theta$, if

$$\lim_{n \rightarrow \infty} \sup_{\theta \in \Theta} d_{BL(\mathbb{D})}(X_n^\theta, X^\theta) = 0.$$

(ii) X_n^θ converges uniformly in probability over Θ to X^θ in \mathbb{D} , and write $X_n^\theta \xrightarrow{P/\Theta} X^\theta$, if

$$\lim_{n \rightarrow \infty} \sup_{\theta \in \Theta} \mathbb{P}(d_{\mathbb{D}}(X_n^\theta, X^\theta) > \varepsilon) = 0$$

for every $\varepsilon > 0$.

If for some $\mu \in \mathcal{M}_1(\mathbb{D})$, it holds that $X_n^\theta \xrightarrow{D/\Theta} X^\theta$ with $X^\theta(\mathbb{P}) = \mu$ for all $\theta \in \Theta$, we also write $X_n^\theta \xrightarrow{D/\Theta} \mu$. Similarly, we may replace the limit random variable X^θ by a point $x \in \mathbb{D}$ by interpreting x as the constant map $(\theta, \omega) \mapsto x$ for $\theta \in \Theta$ and $\omega \in \Omega$.

Note that if the parameter set $\Theta = \{\theta_0\}$ is a singleton, then each type of uniform convergence reduces to the corresponding classical definition of convergence in distribution or probability. If \mathbb{D} is a separable Banach space, we note that Definition B.2 coincides with Definition 3 in [Lundborg, Shah and Peters \(2022\)](#).

PROPOSITION B.3. *Let $(X_n^\theta)_{n \in \mathbb{N}, \theta \in \Theta}$ and $(X^\theta)_{\theta \in \Theta}$ be collections of \mathbb{D} -valued random variables and assume $(X^\theta)_{\theta \in \Theta}$ is separable. Then the following are equivalent:*

- a) $X_n^\theta \xrightarrow{D/\Theta} X^\theta$ as $n \rightarrow \infty$.
- b) For any sequence $(\theta_n)_{n \in \mathbb{N}} \subseteq \Theta$ it holds that $d_{BL}(X_n^{\theta_n}, X^{\theta_n}) \rightarrow 0$ as $n \rightarrow \infty$.
- c) For any sequence $(\theta_n)_{n \in \mathbb{N}} \subseteq \Theta$ there exists a subsequence $(\theta_{k(n)})_{n \in \mathbb{N}}$, with $k: \mathbb{N} \rightarrow \mathbb{N}$ strictly increasing, such that

$$\lim_{k \rightarrow \infty} d_{BL}(X_{k(n)}^{\theta_{k(n)}}, X^{\theta_{k(n)}}) = 0.$$

Moreover, $X_n^\theta \xrightarrow{P/\Theta} X^\theta$ if and only if for any sequence $(\theta_n)_{n \in \mathbb{N}} \subseteq \Theta$ and any $\varepsilon > 0$ it holds that

$$\lim_{n \rightarrow \infty} \mathbb{P}(d_{\mathbb{D}}(X_n^{\theta_n}, X^{\theta_n}) > \varepsilon) = 0.$$

PROOF. This is essentially Lemma 1 in [Kasy \(2019\)](#) for \mathbb{D} -valued random variables, except that we have added the equivalent condition c). The proof for the characterization of uniform convergence in probability is identical to the one given by [Kasy \(2019\)](#), so we focus on the equivalence between a), b), and c). To this end, we to prove that a) \implies b) \implies c) \implies a).

The fact that a) implies b) follows directly from applying the bound

$$d_{BL}(X_n^{\theta_n}, X^{\theta_n}) \leq \sup_{\theta \in \Theta} d_{BL}(X_n^\theta, X^\theta)$$

and taking the limit as $n \rightarrow \infty$. We also see that b) implies c) since any sequence is a subsequence of itself.

We show that c) implies a) by contraposition. Assume the negation of a), that is, there exists an $\varepsilon > 0$ and a sequence $(\theta_n)_{n \in \mathbb{N}} \subseteq \Theta$ such that

$$d_{BL}(X_n^{\theta_n}, X^{\theta_n}) > \varepsilon$$

for all $n \in \mathbb{N}$. Then, for all subsequences $(\theta_{k(n)})$ of (θ_n) , it holds that $d_{BL}(X_{k(n)}^{\theta_{k(n)}}, X^{\theta_{k(n)}})$ does not converge to zero. This implies the negation of c). \square

Proposition B.3 will allow us to extend many results for classical stochastic convergence to uniform stochastic convergence.

COROLLARY B.4. *Let $(X_n^\theta)_{n \in \mathbb{N}, \theta \in \Theta}$ be a collection of \mathbb{D} -valued random variables and let $x \in \mathbb{D}$. Then $X_n^\theta \xrightarrow{\mathcal{D}/\Theta} x$ if and only if $X_n^\theta \xrightarrow{P/\Theta} x$.*

PROOF. For any sequence $(\theta_n)_{n \in \mathbb{N}} \subseteq \Theta$, recall that $X_n^{\theta_n} \xrightarrow{\mathcal{D}} x$ if and only if $X_n^{\theta_n} \xrightarrow{P} x$, see e.g. Lemma 5.1 in [Kallenberg \(2021\)](#). Hence the statement follows directly from Proposition B.3 (combined with Proposition B.1). \square

Our goal is to prove uniform versions of Slutsky's theorem for $D[0, 1]$, Rebolledo's central limit theorem, and the chaining lemma for stochastic processes. To prove Slutsky's lemma for $D[0, 1]$, we first prove a general result for metric spaces.

LEMMA B.5. *Let $(X^\theta, X_n^\theta, Y_n^\theta)_{n \in \mathbb{N}, \theta \in \Theta}$ be a collection of \mathbb{D} -valued random variables and assume that $(X^\theta)_{\theta \in \Theta}$ is separable. If $X_n^\theta \xrightarrow{\mathcal{D}/\Theta} X^\theta$ and $d_{\mathbb{D}}(X_n^\theta, Y_n^\theta) \xrightarrow{P/\Theta} 0$, then it also holds that $Y_n^\theta \xrightarrow{\mathcal{D}/\Theta} X^\theta$.*

PROOF. By the triangle inequality of the bounded Lipschitz metric, we observe that

$$\sup_{\theta \in \Theta} d_{BL}(Y_n^\theta, X^\theta) \leq \sup_{\theta \in \Theta} d_{BL}(Y_n^\theta, X_n^\theta) + \sup_{\theta \in \Theta} d_{BL}(X_n^\theta, X^\theta).$$

The last term converges to zero by the assumption of $X_n^\theta \xrightarrow{\mathcal{D}/\Theta} X^\theta$. For the other term, let $\varepsilon > 0$ and use the partition

$$(d_{\mathbb{D}}(X_n^\theta, Y_n^\theta) \leq \varepsilon) \cup (d_{\mathbb{D}}(X_n^\theta, Y_n^\theta) > \varepsilon)$$

to obtain that

$$\begin{aligned} d_{BL}(X_n^\theta, Y_n^\theta) &= \sup_{f \in BL_1(\mathbb{D})} |\mathbb{E}[f(X_n^\theta) - f(Y_n^\theta)]| \\ &\leq \varepsilon + \sup_{f \in BL_1(\mathbb{D})} \mathbb{E}[|f(X_n^\theta) - f(Y_n^\theta)| \mathbb{1}(d_{\mathbb{D}}(X_n^\theta, Y_n^\theta) > \varepsilon)] \\ &\leq \varepsilon + \mathbb{P}(d_{\mathbb{D}}(X_n^\theta, Y_n^\theta) > \varepsilon). \end{aligned}$$

Taking the supremum over Θ and the limit superior for $n \rightarrow \infty$ finishes the proof. \square

The following formulation of the continuous mapping theorem is analogous to Theorem 1 in [Kasy \(2019\)](#). The proof is almost identical, but we repeat it here for completeness.

PROPOSITION B.6. *Let (\mathbb{D}_1, d_1) and (\mathbb{D}_2, d_2) be metric spaces, and let $\Phi: \mathbb{D}_1 \rightarrow \mathbb{D}_2$ be a Lipschitz continuous map. Let $(X_n^\theta)_{n \in \mathbb{N}, \theta \in \Theta}$ and $(X^\theta)_{\theta \in \Theta}$ be collections of \mathbb{D}_1 -valued random variables, and assume $(X^\theta)_{\theta \in \Theta}$ is separable.*

If $X_n^\theta \xrightarrow{\mathcal{D}/\Theta} X^\theta$ in \mathbb{D}_1 , then $\Phi(X_n^\theta) \xrightarrow{\mathcal{D}/\Theta} \Phi(X^\theta)$ in \mathbb{D}_2 .

PROOF. Note first that if X^θ is in a separable subset $\mathbb{D}_0 \subseteq \mathbb{D}_1$, then the variables $\Phi(X_\theta)$ for $\theta \in \Theta$ are all in the separable subset $\Phi(\mathbb{D}_0) \subseteq \mathbb{D}_2$. Hence it is well-defined to consider uniform convergence in distribution towards $(\Phi(X^\theta))_{\theta \in \Theta}$. Let $f \in BL_1(\mathbb{D}_2)$ and let K be the Lipschitz constant of Φ . Consider the map

$$g: \mathbb{D}_1 \rightarrow \mathbb{R}, \quad g(x) = \min(1, K^{-1})f(\Phi(x)).$$

Then $\|g\|_\infty \leq \|f\|_\infty \leq 1$ and for all $x, y \in \mathbb{D}_1$,

$$\begin{aligned} |g(x) - g(y)| &\leq \min(1, K^{-1})d_2(\Phi(x), \Phi(y)) \\ &\leq \min(1, K^{-1})Kd_1(x, y) \leq d_1(x, y) \end{aligned}$$

Hence $g \in BL_1(\mathbb{D}_1)$. It follows that

$$\begin{aligned} d_{BL_1(\mathbb{D}_2)}(\Phi(X_n^\theta), \Phi(X^\theta)) &= \sup_{f \in BL_1(\mathbb{D}_2)} |\mathbb{E}[f(\Phi(X_n^\theta)) - f(\Phi(X^\theta))]| \\ &\leq \frac{1}{\min(1, K^{-1})} \sup_{g \in BL_1(\mathbb{D}_1)} |\mathbb{E}[g(X_n^\theta) - g(X^\theta)]| \\ &\leq \max(1, K) \cdot d_{BL_1(\mathbb{D}_1)}(X_n^\theta, X^\theta) \end{aligned}$$

Taking the supremum over Θ and the limit superior as $n \rightarrow \infty$ finish the proof. \square

We will also need the following two notions of tightness.

DEFINITION B.7. Let $(\mu^\theta)_{\theta \in \Theta}$ be a family of probability measures on \mathbb{D} , and let $(X^\theta)_{\theta \in \Theta}$ and $(X_n^\theta)_{n \in \mathbb{N}, \theta \in \Theta}$ be collections of \mathbb{D} -valued random variables.

- i) We say that $(\mu^\theta)_{\theta \in \Theta}$ is *tight* if for any $\varepsilon > 0$, there exists a compact set $K \subseteq \mathbb{D}$ such that $\sup_{\theta \in \Theta} \mu^\theta(K^c) < \varepsilon$. We say that $(X^\theta)_{\theta \in \Theta}$ is *uniformly tight* if the collection of distributions $(X^\theta(\mathbb{P}))_{\theta \in \Theta}$ is tight.
- ii) The sequence $((X_n^\theta)_{\theta \in \Theta})_{n \in \mathbb{N}}$ of collections is said to be *sequentially tight* if for any sequence $(\theta_n)_{n \in \mathbb{N}} \subset \Theta$, the sequence of distributions $(X_n^{\theta_n}(\mathbb{P}))_{n \in \mathbb{N}}$ is tight.

Definition B.7 i) is a classical concept, whereas sequential tightness was introduced by [Lundborg, Shah and Peters \(2022\)](#) and relaxes uniform tightness for sequences of variables parametrized over an infinite set.

The importance of tightness is mainly due to Prokhorov's theorem ([Kallenberg, 2021](#), Theorem 23.2), which states that if \mathbb{D} is a Polish space¹, then $(\mu^\theta)_{\theta \in \Theta}$ is tight if and only if all sequences in $(\mu^\theta)_{\theta \in \Theta}$ have a weakly convergent subsequence.

The continuous mapping theorem in Proposition B.6 is more restrictive than the classical theorem as it requires Lipschitz continuity. However, we also have an alternative version of uniform continuous mapping when the limit variable is tight.

PROPOSITION B.8. Let (\mathbb{D}_1, d_1) and (\mathbb{D}_2, d_2) be Polish spaces, and let $(X_n^\theta)_{n \in \mathbb{N}, \theta \in \Theta}$ and $(X^\theta)_{\theta \in \Theta}$ be collections of \mathbb{D}_1 -valued random variables. Assume $(X^\theta)_{\theta \in \Theta}$ is uniformly tight, and let $\Phi: \mathbb{D}_1 \rightarrow \mathbb{D}_2$ be a map that is continuous on the support of $(X^\theta)_{\theta \in \Theta}$.

If $X_n^\theta \xrightarrow{\mathcal{D}/\Theta} X^\theta$ in \mathbb{D}_1 , then $\Phi(X_n^\theta) \xrightarrow{\mathcal{D}/\Theta} \Phi(X^\theta)$ in \mathbb{D}_2 .

PROOF. Same as the proof of Proposition 10 in [Lundborg, Shah and Peters \(2022\)](#), but with norms of differences replaced by metric distances. \square

B.2. Uniform stochastic convergence in Skorokhod space. In this section we consider the special case where $(\mathbb{D}, d_{\mathbb{D}})$ is the Skorokhod space $(D[0, 1], d^\circ)$. We can also equip $D[0, 1]$ with the uniform norm, $\|x\|_\infty = \sup_{t \in [0, 1]} |x_t|$, and it is known that weak convergence based on either $\|\cdot\|_\infty$ or d° are equivalent when the limit is continuous. We now extend this result to stochastic convergence uniformly over Θ .

¹The 'only if' part does not require separability nor completeness.

PROPOSITION B.9 (Skorokhod equivalence). *Let $(X_n^\theta)_{n \in \mathbb{N}, \theta \in \Theta}$ be a collection of $D[0, 1]$ -valued random variables and let $(X^\theta)_{\theta \in \Theta}$ be a uniformly tight collection of $C[0, 1]$ -valued random variables. Then $X_n^\theta \xrightarrow{\mathcal{D}/\Theta} X^\theta$ in $(D[0, 1], d^\circ)$ if and only if $X_n^\theta \xrightarrow{\mathcal{D}/\Theta} X^\theta$ in $(D[0, 1], \|\cdot\|_\infty)$. In the affirmative, $\|X_n^\theta\|_\infty \xrightarrow{\mathcal{D}/\Theta} \|X^\theta\|_\infty$.*

PROOF. To avoid ambiguity in the topology on $D[0, 1]$, we will throughout this proof use \mathbb{D}° to denote the metric space $(D[0, 1], d^\circ)$ and we use \mathbb{D}_∞ to denote the Banach space $(D[0, 1], \|\cdot\|_\infty)$. Note also that $C[0, 1]$ is separable within \mathbb{D}_∞ , so $(X^\theta)_{\theta \in \Theta}$ is separable, and hence the convergence $X_n^\theta \xrightarrow{\mathcal{D}/\Theta} X^\theta$ is well-defined in the non-separable space \mathbb{D}_∞ .

The ‘if’ part is clear since $d^\circ(x, y) \leq \|x - y\|_\infty$ for all $x, y \in D[0, 1]$.

For the ‘only if’ part, assume that $X_n^\theta \xrightarrow{\mathcal{D}/\Theta} X^\theta$ in \mathbb{D}° and let $(\theta_n) \subseteq \Theta$ be an arbitrary sequence. Since $(X^{\theta_n}(\mathbb{P}))$ is tight, Prokhorov’s Theorem asserts that there exists a subsequence $(\theta_{k(n)})$ and a probability distribution μ on $C[0, 1]$ such that $X^{\theta_{k(n)}}(\mathbb{P}) \xrightarrow{wk} \mu$ in \mathbb{D}° . By the triangle inequality

$$d_{BL(\mathbb{D}^\circ)}(X_{k(n)}^{\theta_{k(n)}}, \mu) \leq d_{BL(\mathbb{D}^\circ)}(X_{k(n)}^{\theta_{k(n)}}, X^{\theta_{k(n)}}) + d_{BL(\mathbb{D}^\circ)}(X^{\theta_{k(n)}}, \mu) \rightarrow 0, \quad n \rightarrow \infty.$$

This shows that also $X_{k(n)}^{\theta_{k(n)}}(\mathbb{P}) \xrightarrow{wk} \mu$ in \mathbb{D}° . Now we can use that weak convergence in the Skorokhod topology and the uniform topology are equivalent when the limit is continuous (Kallenberg, 2021, Theorem 23.9 (iii)). We therefore conclude that the convergences $X^{\theta_{k(n)}}(\mathbb{P}) \xrightarrow{wk} \mu$ and $X_{k(n)}^{\theta_{k(n)}}(\mathbb{P}) \xrightarrow{wk} \mu$ also hold in \mathbb{D}_∞ . But then another use of the triangle inequality shows that

$$d_{BL(\mathbb{D}_\infty)}(X_{k(n)}^{\theta_{k(n)}}, X^{\theta_{k(n)}}) \leq d_{BL(\mathbb{D}_\infty)}(X_{k(n)}^{\theta_{k(n)}}, \mu) + d_{BL(\mathbb{D}_\infty)}(\mu, X^{\theta_{k(n)}}) \rightarrow 0.$$

Since $(\theta_{k(n)})$ is a subsequence of the arbitrarily chosen sequence (θ_n) , we conclude that $X_n^\theta \xrightarrow{\mathcal{D}/\Theta} X^\theta$ in \mathbb{D}_∞ by Proposition B.3.

Finally, as the uniform norm is Lipschitz continuous as a map from \mathbb{D}_∞ to \mathbb{R} , the continuous mapping theorem formulated in Proposition B.6 yields that

$$X_n^\theta \xrightarrow{\mathcal{D}/\Theta} X^\theta \text{ in } \mathbb{D}_\infty \quad \implies \quad \|X_n^\theta\|_\infty \xrightarrow{\mathcal{D}/\Theta} \|X^\theta\|_\infty.$$

This establishes the last part of the lemma. \square

Using $\|\mu\|_\infty$ to denote the pushforward measure for any $\mu \in \mathcal{M}_1(D([0, 1]))$ we restate the result above for a fixed limit distribution.

COROLLARY B.10. *Let $(X_n^\theta)_{n \in \mathbb{N}, \theta \in \Theta}$ be a collection of $D[0, 1]$ -valued random variables and let μ be a probability measure on $C[0, 1]$. Then $X_n^\theta \xrightarrow{\mathcal{D}/\Theta} \mu$ in $(D[0, 1], d^\circ)$ if and only if $X_n^\theta \xrightarrow{\mathcal{D}/\Theta} \mu$ in $(D[0, 1], \|\cdot\|_\infty)$. In the affirmative, $\|X_n^\theta\|_\infty \xrightarrow{\mathcal{D}/\Theta} \|\mu\|_\infty$.*

PROOF. Since μ is a probability measure on the Polish space $C[0, 1]$, it is, in particular, tight (Billingsley, 2013, Theorem 1.3). Hence the statement is a special case of Proposition B.9. \square

Now we are ready to prove a uniform version of Slutsky’s theorem in the Skorokhod space.

LEMMA B.11 (Uniform Slutsky in Skorokhod space). *Let $(X^\theta, X_n^\theta, Y_n^\theta)_{n \in \mathbb{N}, \theta \in \Theta}$ be a collection of $D[0, 1]$ -valued random variables such that $Y_n^\theta \xrightarrow{P/\Theta} 0$ and $X_n^\theta \xrightarrow{\mathcal{D}/\Theta} X^\theta$ in $D[0, 1]$. Then it holds that $X_n^\theta + Y_n^\theta \xrightarrow{\mathcal{D}/\Theta} X^\theta$.*

PROOF. Since $Y_n^\theta \xrightarrow{P/\Theta} 0$, Corollary B.10 implies that $\|Y_n^\theta\|_\infty \xrightarrow{D/\Theta} 0$, and Corollary B.4 implies that $\|Y_n^\theta\|_\infty \xrightarrow{P/\Theta} 0$. Using the trivial estimate $d^\circ(x + y, x) \leq \|(x + y) - x\|_\infty = \|y\|_\infty$ for $x, y \in D[0, 1]$, it follows that $d^\circ(X_n^\theta + Y_n^\theta, X_n^\theta) \xrightarrow{P/\Theta} 0$. Combining the latter with $X_n^\theta \xrightarrow{D/\Theta} X^\theta$, the desired conclusion now follows from Lemma B.5. \square

We also have a related result for sums of independent sequences.

LEMMA B.12. *Let $(X_n^\theta, Y_n^\theta)_{n \in \mathbb{N}, \theta \in \Theta}$ be a collection of $D[0, 1]$ -valued random variables and let $(X^\theta)_{\theta \in \Theta}$ and $(Y^\theta)_{\theta \in \Theta}$ be uniformly tight collections of $C[0, 1]$ -valued random variables. Assume that $X_n^\theta \xrightarrow{D/\Theta} X^\theta$ and $Y_n^\theta \xrightarrow{D/\Theta} Y^\theta$ in $D[0, 1]$, and that for each $\theta \in \Theta$ and $n \in \mathbb{N}$, it holds that $X_n^\theta \perp\!\!\!\perp Y_n^\theta$. Let Z^θ have distribution $X^\theta(\mathbb{P}) * Y^\theta(\mathbb{P})$, that is, the same distribution as the sum of two independent copies of each of X^θ and Y^θ .*

Then it also holds that $X_n^\theta + Y_n^\theta \xrightarrow{D/\Theta} Z^\theta$ in $(D[0, 1], \|\cdot\|_\infty)$.

PROOF. We may assume without loss of generality that $X^\theta \perp\!\!\!\perp Y^\theta$ and that $Z^\theta = X^\theta + Y^\theta$. Let $(\theta_n) \subseteq \Theta$ be an arbitrary sequence. By tightness of $(X^\theta)_{\theta \in \Theta}$ and $(Y^\theta)_{\theta \in \Theta}$, we can apply Prokhorov's theorem twice to obtain probability measures μ and ν on $C[0, 1]$, and a subsequence $(\theta_{k(n)})$, such that $X^{\theta_{k(n)}}(\mathbb{P}) \xrightarrow{wk} \mu$ and $Y^{\theta_{k(n)}}(\mathbb{P}) \xrightarrow{wk} \nu$. Hence the product measures converge,

$$X^{\theta_{k(n)}}(\mathbb{P}) \otimes Y^{\theta_{k(n)}}(\mathbb{P}) \xrightarrow{wk} \mu \otimes \nu,$$

in $C[0, 1]$ as $n \rightarrow \infty$, see, for example, Theorem 2.8 (ii) in Billingsley (2013).

Since $X_n^\theta \xrightarrow{D/\Theta} X^\theta$ in $D[0, 1]$ by assumption and (X^θ) is uniformly tight in $C[0, 1]$, Proposition B.9 implies that the convergence also holds in $(D[0, 1], \|\cdot\|_\infty)$. The triangle inequality now yields

$$d_{BL}(X_{k(n)}^{\theta_{k(n)}}, \mu) \leq d_{BL}(X_{k(n)}^{\theta_{k(n)}}, X^{\theta_{k(n)}}) + d_{BL}(X^{\theta_{k(n)}}, \mu) \rightarrow 0,$$

so also $X_{k(n)}^{\theta_{k(n)}}(\mathbb{P}) \xrightarrow{wk} \mu$ in $(D[0, 1], \|\cdot\|_\infty)$. An analogous computation shows that $Y_{k(n)}^{\theta_{k(n)}}(\mathbb{P}) \xrightarrow{wk} \nu$, and hence also

$$X_{k(n)}^{\theta_{k(n)}}(\mathbb{P}) \otimes Y_{k(n)}^{\theta_{k(n)}}(\mathbb{P}) \xrightarrow{wk} \mu \otimes \nu$$

in the product space $D[0, 1] \times D[0, 1]$ endowed with the uniform product topology. From the independence statements $X^\theta \perp\!\!\!\perp Y^\theta$ and $X_n^\theta \perp\!\!\!\perp Y_n^\theta$, we have thus shown that

$$(X^{\theta_{k(n)}}, Y^{\theta_{k(n)}}) \xrightarrow{\mathcal{D}} \mu \otimes \nu \quad \text{and} \quad (X_{k(n)}^{\theta_{k(n)}}, Y_{k(n)}^{\theta_{k(n)}}) \xrightarrow{\mathcal{D}} \mu \otimes \nu$$

in the uniform product topology. Since addition $+$: $D[0, 1] \times D[0, 1] \rightarrow D[0, 1]$ is continuous with respect to this topology, we conclude by the classical continuous mapping theorem that

$$Z^{\theta_{k(n)}} = X_{k(n)}^{\theta_{k(n)}} + Y_{k(n)}^{\theta_{k(n)}} \xrightarrow{\mathcal{D}} \mu * \nu \quad \text{and} \quad X_{k(n)}^{\theta_{k(n)}} + Y_{k(n)}^{\theta_{k(n)}} \xrightarrow{\mathcal{D}} \mu * \nu.$$

It now follows that

$$\begin{aligned} & d_{BL}(X_{k(n)}^{\theta_{k(n)}} + Y_{k(n)}^{\theta_{k(n)}}, Z^{\theta_{k(n)}}) \\ & \leq d_{BL}(X_{k(n)}^{\theta_{k(n)}} + Y_{k(n)}^{\theta_{k(n)}}, \mu * \nu) + d_{BL}(\mu * \nu, Z^{\theta_{k(n)}}) \rightarrow 0. \end{aligned}$$

Since $(\theta_{k(n)})$ is a subsequence of the arbitrarily chosen sequence (θ_n) , we conclude that $X_n^\theta + Y_n^\theta \xrightarrow{D/\Theta} Z^\theta$ in $(D[0, 1], \|\cdot\|_\infty)$ by Proposition B.3. \square

We also need the following lemma, which is a generalization of the classical result: pointwise convergence of a sequence of monotone functions towards a continuous limit is in fact uniform over compact intervals.

LEMMA B.13. *Let $(X_n^\theta)_{n \in \mathbb{N}, \theta \in \Theta}$ be a collection of $D[0, 1]$ -valued random variables with non-decreasing sample paths. Let $(f^\theta)_{\theta \in \Theta} \subset C[0, 1]$ be a uniformly equicontinuous collection of non-decreasing functions. If $X_n^\theta(t) \xrightarrow{P/\Theta} f^\theta(t)$ for each $t \in [0, 1]$, then it also holds that*

$$\sup_{t \in [0, 1]} |X_n^\theta(t) - f^\theta(t)| \xrightarrow{P/\Theta} 0.$$

PROOF. Let $\varepsilon > 0$. By uniform equicontinuity we can find $0 = t_1 < \dots < t_k = 1$ such that $f^\theta(t_i) - f^\theta(t_{i-1}) < \varepsilon/2$ for all θ and i . Using that X_n^θ and f^θ are non-decreasing, we observe that for $t_{i-1} \leq t \leq t_i$:

$$X_n^\theta(t) - f^\theta(t) \leq X_n^\theta(t_i) - f^\theta(t_i) + \varepsilon/2,$$

$$X_n^\theta(t) - f^\theta(t) \geq X_n^\theta(t_{i-1}) - f^\theta(t_{i-1}) - \varepsilon/2.$$

Combining the inequalities over the entire grid we have

$$\sup_{t \in [0, 1]} |X_n^\theta(t) - f^\theta(t)| \leq \max_{i=0, \dots, k} |X_n^\theta(t_i) - f^\theta(t_i)| + \varepsilon/2.$$

By assumption, $X_n^\theta(t) \xrightarrow{P/\Theta} f^\theta(t)$ for each t , and in particular

$$\max_{i=0, \dots, k} |X_n^\theta(t_i) - f^\theta(t_i)| \xrightarrow{P/\Theta} 0$$

as $n \rightarrow \infty$. We therefore conclude that

$$\sup_{\theta \in \Theta} \mathbb{P} \left(\sup_{t \in [0, 1]} |X_n^\theta(t) - f^\theta(t)| > \varepsilon \right) \leq \sup_{\theta \in \Theta} \mathbb{P} \left(\max_{i=0, \dots, k} |X_n^\theta(t_i) - f^\theta(t_i)| > \varepsilon/2 \right) \rightarrow 0$$

as $n \rightarrow \infty$. □

The last auxiliary result of this section is an example of Prokhorov's method of "tightness + identification of limit".

LEMMA B.14. *Let $(\mathbb{D}, d_{\mathbb{D}})$ be either $(C[0, 1], \|\cdot\|_\infty)$ or $(D[0, 1], d^\circ)$, and let $(X_n^\theta)_{n \in \mathbb{N}, \theta \in \Theta}$ be a collection of \mathbb{D} -valued random variables with $(X_n^\theta)_{\theta \in \Theta}$ separable. Suppose that*

- *The finite dimensional marginals converge uniformly: for any $0 \leq t_1 < \dots < t_k \leq 1$*

$$\pi_{t_1, \dots, t_k}(X_n^\theta) \xrightarrow{D/\Theta} \pi_{t_1, \dots, t_k}(X^\theta), \quad n \rightarrow \infty,$$

where $\pi_{t_1, \dots, t_k} : \mathbb{D} \rightarrow \mathbb{R}^k$ is the projection given by $\pi_{t_1, \dots, t_k}(x) = (x(t_1), \dots, x(t_k))$.

- $(X_n^\theta)_{n \in \mathbb{N}, \theta \in \Theta}$ *is sequentially tight.*
- $(X_n^\theta)_{n \in \mathbb{N}, \theta \in \Theta}$ *is uniformly tight.*

Then $X_n^\theta \xrightarrow{D/\Theta} X^\theta$ as $n \rightarrow \infty$.

PROOF. The statement is analogous to Proposition 18 in [Lundborg, Shah and Peters \(2022\)](#), the difference being that the functionals $\langle \cdot, h \rangle$ in [Lundborg, Shah and Peters \(2022\)](#) have been replaced by the functionals π_{t_1, \dots, t_k} .

The proof of [Lundborg, Shah and Peters \(2022\)](#) also works in our case, given that the finite dimensional marginals form a separating class for the both the Borel algebra on $C[0, 1]$ and the Borel algebra on $D[0, 1]$. This is established in [Billingsley \(2013\)](#), Example 1.3 and Theorem 12.5 (iii). □

B.3. Chaining in time uniformly over a parameter. We extend the basic chaining arguments to hold uniformly over Θ . Our arguments closely follow those of Pollard (1984, Chapter VII.2.) and Newey (1991). The results are formulated for processes indexed over a general metric space T , but we will only apply the results in the case $T = [0, 1]$. We have the following extension of stochastic equicontinuity to the uniform setting.

DEFINITION B.15. A collection of sequences

$$(Z^{(n),\theta})_{n \in \mathbb{N}, \theta \in \Theta} = \left(Z_t^{(n),\theta} \right)_{t \in T, n \in \mathbb{N}, \theta \in \Theta}$$

of stochastic processes indexed over a metric space (T, d) is called *stochastically equicontinuous uniformly over Θ* if for all $\varepsilon, \eta > 0$ there exists $\delta > 0$ such that

$$\limsup_{n \rightarrow \infty} \sup_{\theta \in \Theta} \mathbb{P} \left(\sup_{s, t \in T: d(s, t) \leq \delta} |Z_s^{(n),\theta} - Z_t^{(n),\theta}| > \varepsilon \right) < \eta.$$

In Section 2.8.2 of van der Vaart and Wellner (1996), the same definition is given in the context of empirical processes. Recall that we write, e.g., $Z^{(n)}$ as a shorthand for $Z^{(n),\theta}$ and let the dependency on θ be implicit for notational ease. We also write $\sup_{d(s, t) \leq \delta}$ as a shorthand for $\sup_{s, t \in T: d(s, t) \leq \delta}$. Definition B.15 is a direct extension of pointwise stochastic equicontinuity. Accordingly, Theorem 2.1 from Newey (1991) generalizes as follows:

LEMMA B.16. Let $(Z_t^{(n)})_{t \in T, n \in \mathbb{N}}$ be a sequence of stochastic processes indexed by a compact metric space T . Assume that $(Z_t^{(n)})$ is stochastically equicontinuous uniformly over Θ and that for each $t \in T$ it holds that $Z_t^{(n)} \xrightarrow{P/\Theta} 0$. Then $\sup_{t \in T} |Z_t^{(n)}| \xrightarrow{P/\Theta} 0$ as $n \rightarrow \infty$.

PROOF. Let $\varepsilon, \eta > 0$ be given, and let $\delta > 0$ be the corresponding distance obtained from the uniform stochastic equicontinuity of $(Z^{(n)})$. By compactness of T there exists a finite set $T^* \subseteq T$ such that $T = \bigcup_{t \in T^*} B(t, \delta)$. By the triangle inequality we get that

$$\sup_{t \in T} |Z_t^{(n)}| = \sup_{t \in T^*} \sup_{s \in B(t, \delta)} |Z_s^{(n)}| \leq \sup_{t \in T^*} |Z_t^{(n)}| + \sup_{t \in T^*} \sup_{s \in B(t, \delta)} |Z_s^{(n)} - Z_t^{(n)}|.$$

Since T^* is finite, it follows that $\sup_{t \in T^*} |Z_t^{(n)}| \xrightarrow{P/\Theta} 0$, which combined with the inequality implies that

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \sup_{\theta \in \Theta} \mathbb{P} \left(\sup_{t \in T} |Z_t^{(n)}| > 2\varepsilon \right) \\ & \leq 0 + \limsup_{n \rightarrow \infty} \sup_{\theta \in \Theta} \mathbb{P} \left(\sup_{t \in T^*} \sup_{s \in B(t, \delta)} |Z_t^{(n)} - Z_s^{(n)}| > \varepsilon \right) \leq \eta. \end{aligned}$$

As $\varepsilon, \eta > 0$ were chosen arbitrarily, we conclude that $\sup_{t \in T} |Z_t^{(n)}| \xrightarrow{P/\Theta} 0$. \square

To establish uniform stochastic equicontinuity we extend the chaining lemma to a uniform setting. To formulate the theorem we first need some classical definitions related to chaining.

DEFINITION B.17. Let T be a compact metric space. A subset $T^* \subseteq T$ is called a δ -net if $\bigcup_{t \in T^*} B(t, \delta) = T$. The covering number

$$N(\delta) = N(\delta, T) := \min \{ |T^*| : T^* \subseteq T, T^* \text{ is a } \delta\text{-net} \}$$

is the smallest possible cardinality of a δ -net, which is finite by compactness. The associated covering integral is

$$J(\delta) = \int_0^\delta (2 \log(N(\varepsilon)/\varepsilon))^{\frac{1}{2}} d\varepsilon, \quad 0 \leq \delta \leq 1.$$

LEMMA B.18. *Let (T, d) be a metric space with finite covering integral $J(\cdot)$ and let $(Z_t^\theta)_{t \in T, \theta \in \Theta}$ be a collection of stochastic processes indexed by T with continuous sample paths. Assume there is a uniform constant $\varsigma > 0$ such that, for all $s, t \in T$ and $\eta > 0$,*

$$\sup_{\theta \in \Theta} \mathbb{P} \left(|Z_s^\theta - Z_t^\theta| > \eta \cdot d(s, t) \right) \leq 2e^{-\frac{\eta^2}{2\varsigma^2}}.$$

Then, for all $0 < \varepsilon < 1$,

$$\sup_{\theta \in \Theta} \mathbb{P} \left(\sup_{d(s, t) \leq \varepsilon} |Z_s^\theta - Z_t^\theta| > 26\varsigma J(\varepsilon) \right) \leq 2\varepsilon.$$

PROOF. The lemma is a direct consequence of classical chaining lemma (Pollard, 1984, page 144). For each $\theta \in \Theta$, the conditions of the chaining lemma are met for $(Z_t^\theta)_{t \in T}$ with sub-exponential factor ς . This implies, in particular, that for any $\theta \in \Theta$ and $0 < \varepsilon < 1$,

$$\mathbb{P} \left(\sup_{d(s, t) \leq \varepsilon} |Z_s^\theta - Z_t^\theta| > 26\varsigma J(\varepsilon) \right) \leq 2\varepsilon,$$

which is equivalent to the conclusion of the lemma. \square

This immediately implies the following corollary.

COROLLARY B.19. *Let (T, d) be a metric space with finite covering integral $J(\cdot)$ and let $(Z^{(n), \theta})$ be a sequence of stochastic processes on T with continuous sample paths. Assume there exists a constant $\varsigma > 0$ such that, for all $s, t \in T$ and $\eta > 0$ and $n \in \mathbb{N}$,*

$$\sup_{\theta \in \Theta} \mathbb{P} \left(|Z_s^{(n), \theta} - Z_t^{(n), \theta}| > \eta \cdot d(s, t) \right) \leq 2e^{-\frac{\eta^2}{2\varsigma^2}}.$$

Then $(Z^{(n)})$ is stochastically equicontinuous uniformly over Θ .

For stochastic processes with continuous sample paths, stochastic equicontinuity turns out to be equivalent to sequential tightness (Definition B.7 ii).

PROPOSITION B.20. *Let $(Z^{(n), \theta})_{n \in \mathbb{N}, \theta \in \Theta}$ be a collection of $C[0, 1]$ -valued random variables such that $\mathbb{P}(Z_0^{(n), \theta} = 0) = 1$ all $n \in \mathbb{N}$ and $\theta \in \Theta$. The following are equivalent:*

1. $(Z^{(n), \theta})$ is stochastically equicontinuous uniformly over Θ .
2. $(Z^{(n), \theta})$ is sequentially tight.

PROOF. The equivalence is a straightforward application of Theorem 7.3 in Billingsley (2013). Condition (i) of the aforementioned theorem is satisfied for any sequence of measures from the collection $(Z^{(n), \theta}(\mathbb{P}))_{n \in \mathbb{N}, \theta \in \Theta}$, since $Z_0^{(n), \theta} = 0$ almost surely for all n and θ . For any sequence $(\theta_n) \subseteq \Theta$, stochastic equicontinuity uniformly over Θ implies condition (ii) of Theorem 7.3 in Billingsley (2013) for the measures $((Z^{(n), \theta_n})(\mathbb{P}))$. We therefore conclude that stochastic equicontinuity uniformly over Θ implies sequential tightness.

On the contrary, assume that $(Z^{(n), \theta})$ is sequentially tight and let $\varepsilon, \eta > 0$ be given. For each n , choose θ_n such that

$$\sup_{\theta \in \Theta} \mathbb{P} \left(\sup_{|s-t| \leq \delta} |Z_s^{(n), \theta} - Z_t^{(n), \theta}| \geq \varepsilon \right) \leq \mathbb{P} \left(\sup_{|s-t| \leq \delta} |Z_s^{(n), \theta_n} - Z_t^{(n), \theta_n}| \geq \varepsilon \right) + \frac{1}{n}.$$

Since $((Z^{(n), \theta_n})(\mathbb{P}))$ is tight by assumption, condition (ii) of Theorem 7.3 asserts that there exists $\delta, N > 0$ such that

$$\mathbb{P} \left(\sup_{|s-t| \leq \delta} |Z_s^{(n), \theta_n} - Z_t^{(n), \theta_n}| \geq \varepsilon \right) < \eta$$

for $n \geq N$. Combining both inequalities and taking the limit superior finish the proof. \square

C. The Functional Martingale CLT. In this section we state Rebolledo's martingale CLT (Rebolledo, 1980) based on its formulation in Andersen et al. (1993), and then we extend the result to a uniform version without fixed variance functions. The one-dimensional case suffices for our purpose, so for simplicity, every local martingale in the following is a real-valued stochastic process. For a local square integrable martingale (M_t) , we let $\langle M \rangle(t)$ denote its quadratic characteristic. The theorem requires a condition on the jumps of the local martingales, for which we will need the following definition.

DEFINITION C.1. Let M_t be a local square integrable \mathcal{F}_t -martingale. For any $\varepsilon > 0$, we define $\langle M_\varepsilon \rangle(t)$ to be the quadratic characteristic of the pure jump-process given by

$$t \mapsto \sum_{0 \leq s \leq t} M_s \mathbb{1}(|\Delta M_s| > \varepsilon).$$

We also need a representation of Gaussian martingales, which ensures their continuity.

PROPOSITION C.2. Let $(B_t)_{t \in [0, \infty)}$ be a Brownian motion on $[0, \infty)$ with continuous sample paths. For every non-decreasing $f \in C[0, 1]$, the process $(B_{f(t)})_{t \in [0, 1]}$ is a continuous mean zero Gaussian martingale on $[0, 1]$ with variance function f .

Consequently, if $U = (U_t)_{t \in [0, 1]}$ is a mean zero Gaussian martingale with a continuous variance function V , then U has the distributional representation

$$(10) \quad (U_t)_{t \in [0, 1]} \stackrel{\mathcal{D}}{=} (B_{V(t)})_{t \in [0, 1]}.$$

PROOF. Let $f \in C[0, 1]$ be non-decreasing. From the properties of Brownian motion, it follows directly that the time-transformed process $(B_{f(t)})_{t \in [0, 1]}$ is a mean zero Gaussian process with variance function f . Since f is continuous, each sample path $t \mapsto B_{f(t)}$ is a composition of continuous functions and thus continuous itself. Since f is non-decreasing, the time-transformation also preserves the martingale property. This establishes the first part.

For the second part, recall that the covariance function of a martingale is determined by its variance function. Hence the first part implies that the right-hand side in (10) is a Gaussian process with the same mean and covariance structure as the left-hand side. Since the distribution of a Gaussian processes is uniquely determined by its mean and covariance structure, the equality in distribution follows. \square

Proposition C.2 is a simple, distributional variant of the Dubins-Schwarz theorem, see Revuz and Yor (2013), Chapter V, Theorems 1.6 and 1.7. The Dubins-Schwarz theorem implies that, in fact, $U_t = B_{V(t)}$ for $t \in [0, 1]$, where B is a Brownian motion on $[0, V(1)]$. For the purpose of this work we only need the simpler, distributional equality (10).

We can now formulate Rebolledo's CLT for local martingales. To this end, note that Proposition C.2 ensures the existence of the continuous Gaussian limit martingale U when the variance function V is continuous.

THEOREM C.3 (Rebolledo's CLT). Let $(U^{(n)})_{n \in \mathbb{N}}$ be a sequence a local square integrable martingales in $D[0, 1]$, possibly defined on different sample spaces and with different filtrations for each $n \in \mathbb{N}$. Let U be a continuous Gaussian martingale with continuous variance function $V : [0, 1] \rightarrow [0, \infty)$, and assume that $U_0^{(n)} = U_0 = 0$. Suppose that for every $t \in [0, 1]$ and $\varepsilon > 0$,

$$\langle U^{(n)} \rangle(t) \xrightarrow{P} V(t) \quad \text{and} \quad \langle U_\varepsilon^{(n)} \rangle(t) \xrightarrow{P} 0,$$

as $n \rightarrow \infty$. Then it holds that $U^{(n)} \xrightarrow{\mathcal{D}} U$ in $D[0, 1]$ as $n \rightarrow \infty$.

PROOF. This is a special case of Theorem II.5.2 in [Andersen et al. \(1993\)](#). \square

The general formulation of Rebolledo's CLT above, which allows for n -dependent sample spaces and filtrations, can now be leveraged to obtain a uniform version via the sequential characterization of uniform stochastic convergence.

THEOREM C.4 (Uniform Rebolledo CLT). *For each $n \in \mathbb{N}$ and $\theta \in \Theta$:*

- *Let $\mathcal{F}^{(n),\theta} = (\mathcal{F}_t^{(n),\theta})_{t \in [0,1]}$ be a filtration satisfying the usual conditions.*
- *Let $U_t^{(n),\theta}$ be a local square integrable $\mathcal{F}_t^{(n),\theta}$ -martingale in $D[0,1]$ with $U_0^{(n),\theta} = 0$.*
- *Let $V^\theta: [0,1] \rightarrow [0,\infty)$ be a non-decreasing function with $V^\theta(0) = 0$.*

Assume that $(V^\theta)_{\theta \in \Theta}$ is uniformly equicontinuous and that $\sup_{\theta \in \Theta} V^\theta(1) < \infty$. Assume further that for every $\varepsilon > 0$ and $t \in [0,1]$,

$$(11) \quad \langle U^{(n),\theta} \rangle(t) \xrightarrow{P/\Theta} V^\theta(t) \quad \text{and} \quad \langle U_\varepsilon^{(n),\theta} \rangle(t) \xrightarrow{P/\Theta} 0,$$

as $n \rightarrow \infty$. Then it holds that

$$U^{(n),\theta} \xrightarrow{\mathcal{D}/\Theta} U^\theta, \quad n \rightarrow \infty,$$

in $D[0,1]$ uniformly over Θ , where for each $\theta \in \Theta$, U^θ is a mean zero continuous Gaussian martingale on $[0,1]$ with variance function V^θ .

PROOF. We will use the characterization of uniform convergence as stated in Proposition B.3 c). To this end, let $(\theta_n) \subseteq \Theta$ be an arbitrary sequence. By assumption $(V_{\theta_n})_{n \in \mathbb{N}}$ is a uniformly equicontinuous and bounded sequence of functions on a compact interval, so the Arzelà–Ascoli theorem states that there exists a subsequence $\theta_{k(n)}$, with $k: \mathbb{N} \rightarrow \mathbb{N}$ strictly increasing, and a function $\tilde{V} \in C[0,1]$ such that

$$\sup_{t \in [0,1]} |V^{\theta_{k(n)}}(t) - \tilde{V}(t)| \rightarrow 0, \quad n \rightarrow \infty.$$

Since each function $V^{\theta_{k(n)}}$ is non-decreasing, it follows that \tilde{V} is non-decreasing. It also holds that $\tilde{V}(0) = \lim_{n \rightarrow \infty} V^{\theta_{k(n)}}(0) = 0$, and therefore \tilde{V} is the variance function of a continuous Gaussian martingale \tilde{U} with $\tilde{U}_0 = 0$.

By assumption of the convergences in (11), we may conclude that

$$|\langle U^{(k(n)),\theta_{k(n)}} \rangle(t) - \tilde{V}(t)| \leq \underbrace{|\langle U^{(k(n)),\theta_{k(n)}} \rangle(t) - V^{\theta_{k(n)}}(t)|}_{\xrightarrow{P} 0} + \underbrace{|V^{\theta_{k(n)}}(t) - \tilde{V}(t)|}_{\rightarrow 0} \xrightarrow{P} 0$$

and that $\langle U_\varepsilon^{(k(n)),\theta_{k(n)}} \rangle(t) \rightarrow 0$ as $n \rightarrow \infty$. Thus we have established the conditions of the classical Rebolledo CLT – Theorem C.3 – for the sequence $U^{(k(n)),\theta_{k(n)}}$ and the Gaussian martingale \tilde{U} with variance function \tilde{V} . We therefore conclude that

$$U^{(k(n)),\theta_{k(n)}} \xrightarrow{\mathcal{D}} \tilde{U}$$

in $D[0,1]$ as $n \rightarrow \infty$.

We now establish that the sequence $(U^{\theta_{k(n)}})$ also converges in distribution to \tilde{U} in $C[0,1]$, and in particular also in $D[0,1]$. To this end, we use the characterization of convergence in distribution in $C[0,1]$ from Theorem 7.5 in [Billingsley \(2013\)](#), which states that we need to show that

1. For all $0 \leq t_1 < \dots < t_m \leq 1$, it holds that

$$(U_{t_1}^{\theta_{k(n)}}, \dots, U_{t_m}^{\theta_{k(n)}}) \xrightarrow{\mathcal{D}} (\tilde{U}_{t_1}, \dots, \tilde{U}_{t_m}), \quad n \rightarrow \infty.$$

2. For all $\varepsilon > 0$

$$\lim_{\delta \rightarrow 0^+} \limsup_{n \rightarrow \infty} \mathbb{P} \left(\sup_{|t-s| < \delta} |U_t^{\theta_{k(n)}} - U_s^{\theta_{k(n)}}| > \varepsilon \right) = 0.$$

The first condition is clear since all the marginals are multivariate Gaussian, and the mean and variance of the sequence converges to the mean and variance of the limit distribution. The second condition follows from the same computation as in the proof of Lemma A.11. By Theorem 7.5 in Billingsley (2013) we therefore conclude that

$$U^{\theta_{k(n)}} \xrightarrow{\mathcal{D}} \tilde{U}, \quad \text{for } n \rightarrow \infty,$$

in $C[0, 1]$, and hence also in $D[0, 1]$.

We can now apply the triangle inequality for the bounded Lipschitz metric to conclude that

$$d_{BL}(U^{(k(n)), \theta_{k(n)}}, U^{\theta_{k(n)}}) \leq d_{BL}(U^{(k(n)), \theta_{k(n)}}, \tilde{U}) + d_{BL}(\tilde{U}, U^{\theta_{k(n)}}) \longrightarrow 0.$$

Since $(\theta_n) \subseteq \Theta$ was an arbitrary sequence, we conclude that $U^{(n), \theta} \xrightarrow{\mathcal{D}/\Theta} U^\theta$ by Proposition B.3. \square

The following proposition gives explicit expressions for the quadratic characteristics that appear in Rebolledo's CLT in the special case where the local martingales are given as stochastic integrals with respect to a compensated counting processes.

PROPOSITION C.5. *Let N_1, \dots, N_n be counting processes and assume that for each $j = 1, \dots, n$, N_j has an absolutely continuous $\mathcal{F}_t^{(n)}$ -compensator $\Lambda_{j,t}$ such that $M_{j,t} = N_{j,t} - \Lambda_{j,t}$ is a locally square integrable $\mathcal{F}_t^{(n)}$ -martingale. Let H_1, \dots, H_n be locally bounded $\mathcal{F}_t^{(n)}$ -predictable processes, and define the process $U_t^{(n)} = \sum_{j=1}^n \int_0^t H_{j,s} dM_{j,s}$. Then $U_t^{(n)}$ is a local square integrable $\mathcal{F}_t^{(n)}$ -martingale, and for any $t, \varepsilon > 0$ it holds that*

$$\begin{aligned} \langle U^{(n)} \rangle(t) &= \sum_{j=1}^n \int_0^t H_{j,s}^2 d\Lambda_{j,s}, \\ \langle U_\varepsilon^{(n)} \rangle(t) &= \sum_{j=1}^n \int_0^t H_{j,s}^2 \mathbb{1}(|H_{j,s}| \geq \varepsilon) d\Lambda_{j,s}. \end{aligned}$$

PROOF. See the discussion following Theorem II.5.2 in Andersen et al. (1993), in particular equations (2.5.6) and (2.5.8). \square

D. Estimation of λ and G . The asymptotic theory for estimation of the LCM crucially relies on $\hat{\lambda}^{(n)}$ and $\hat{G}^{(n)}$ being consistent, and more importantly, having a product error decaying at an $n^{-1/2}$ -rate. Therefore, a central question when applying the test, is how to model λ and G .

In principle, we could use parametric models to learn $\hat{\lambda}^{(n)}$ and $\hat{G}^{(n)}$, and under such models it should be possible to achieve $n^{-1/2}$ -rates. For example, if we consider a parametrization $(t, \theta) \mapsto \lambda_t(\theta)$ which is $\kappa(t)$ -Lipschitz in $\theta \in \Theta \subseteq \mathbb{R}^p$ for each t , then

$$h(n)^2 = E \left(\int_0^1 (\lambda_t(\theta_0) - \lambda_t(\hat{\theta}^{(n)}))^2 dt \right) \leq \|\kappa\|_{L_2([0,1])}^2 E \|\theta_0 - \hat{\theta}^{(n)}\|_{\mathbb{R}^p}^2.$$

Thus the rates from parametric asymptotic theory can be converted to rates for g and h .

However, it is of greater interest if sufficient rates can be achieved with nonparametric estimators. Below we give concrete examples of nonparametric models and discuss which rates are achievable. For simplicity, we focus on the case where $\mathcal{F}_t = \mathcal{F}_t^{N,Z}$ and where $G_t = X_t - \Pi_t$ as in the introductory example.

D.1. Nonparametric functional estimation of Π . As seen in Section 6.1, assumptions on the form of Π turn the general estimation problem into a concrete problem of estimating a function.

If the system is Markovian, it can be reasonable to assume a *functional concurrent model*. The model asserts that $\Pi_t = \mu(t, Z_t)$ for a bivariate function μ , and a survey of methods for estimating μ is given by Maity (2017). Notably, Jiang and Wang (2011) achieve an $n^{-1/3}$ -rate of $g(n)$ under certain regularity and moment assumptions, see their Theorem 3.3. That result also holds if Z is replaced by a linear predictor $\beta^T \mathbf{Z}$ of several covariates.

Consider again the historical linear regression model from Section 6.1, and assume that the effect of Z on X is homogeneous over time. That is, $\rho_X(s, t) = \tilde{\rho}_X(t - s)$ for some function $\tilde{\rho}_X$. This submodel is known as the *functional convolution model*, since Π can be written as the convolution of Z and $\tilde{\rho}_X$. Applying the Fourier transform converts it into a (complex) linear concurrent model, so by Plancherel's theorem one can leverage the convergence rates from the concurrent model. Manrique (2016, Theorem 16) uses this idea to transfer the $n^{-1/4}$ -rate of the *functional ridge regression estimator* (Manrique, Crambes and Hilgert, 2018) to the convolution model, which holds under modest moment conditions on the data. With additional distributional assumptions, we conjecture that faster rate results for the linear concurrent model can also be leveraged to the convolution model. Şentürk and Müller (2010) consider a similar model under the assumption that

$$\rho_X(s, t) = \mathbb{1}(t - \Delta \leq s \leq t) \tilde{\rho}_X^1(t) \tilde{\rho}_X^2(t - s),$$

for two functions $\tilde{\rho}_X^1$ and $\tilde{\rho}_X^2$ and a lag $\Delta > 0$. They establish a pointwise rate result for the response curve, but it is not obvious how to cast their result as a polynomial rate for $g(n)$.

For the full historical functional linear model we are not aware of any published rate results. Yuan and Cai (2010); Cai and Yuan (2012) establish rates on the prediction error for scalar-on-function regression, and Yao, Müller and Wang (2005) establish various rates for function-on-function regression, but in a non-historical setting. Based on the former, we give a heuristic for which rates are achievable for $g(n)$ in this model. If $\hat{\Pi}$ is based on a kernel estimate $\hat{\rho}_X^{(n)}$ of ρ_X , then Tonelli's theorem yields

$$\begin{aligned} g(n)^2 &= \left\| \Pi - \hat{\Pi}^{(n)} \right\|_2^2 = \mathbb{E} \left(\int_0^1 \left(\int_0^t (\rho_X(s, t) - \hat{\rho}_X^{(n)}(s, t)) Z_s ds \right)^2 dt \right) \\ &= \int_0^1 \mathbb{E} \left(\left(\int_0^t (\rho_X(s, t) - \hat{\rho}_X^{(n)}(s, t)) Z_s ds \right)^2 \right) dt. \end{aligned}$$

Theorem 4 in Cai and Yuan (2012) asserts that we, under certain regularity conditions, can estimate $\rho_X(\cdot, t)$ such that

$$\mathbb{E} \left(\left(\int_0^t (\rho_X(s, t) - \hat{\rho}_X^{(n)}(s, t)) Z_s ds \right)^2 \right)$$

decays at a $n^{-2r_t/(2r_t+1)}$ -rate for a fixed t . Here r_t is a constant describing the eigenvalue decay of a certain operator related to the autocovariance of Z and the regularity of ρ_X . As a

concrete example, if Z is a Wiener process and $\rho_X(\cdot, t) \in \mathcal{W}_2^m([0, t])$ is in the m -th Sobolev space for each $t > 0$, then $r_t = 1 + m$ and $g(n)$ will converge at an $n^{-(1+m)/(2m+3)}$ -rate, see the discussion after Corollary 8 in [Yuan and Cai \(2010\)](#). Based on these arguments, we believe that the desired $n^{-(1/4+\varepsilon)}$ -rate for $g(n)$ is achievable with suitable regularity assumptions on Z and ρ_X .

D.2. Estimation of λ . Within the framework of the Cox model, [Wells \(1994\)](#) demonstrate that the baseline intensity can be estimated with rate $n^{-2/5}$ using a standard kernel smoothing technique. With the parametric $n^{-1/2}$ -rate on the remaining parameters, this translates readily into $h(n) = O(n^{-2/5})$.

We suspect that the same rate should also be attainable in a sparse setting with high-dimensional covariates, for example by applying the smoothing approach of [Wells \(1994\)](#) to the baseline hazard estimators of e.g. [Fang, Ning and Liu \(2017\)](#) and [Hou, Bradic and Xu \(2023\)](#).

[Hiabu et al. \(2021\)](#) consider the more general multiplicative intensity model with $\lambda_t = \mathbb{1}(T \geq t)f(t, Z_t)$, where f has a multiplicative structure over its arguments. They introduce an estimator with optimal rate $h(n) = n^{-2/(5+d)}$, where d is the dimension of Z . For $d > 3$, we therefore need faster rates on $g(n)$ in order for the LCT to maintain type I error control.

Omitting the multiplicative structure on f , [Bender et al. \(2020\)](#) propose a general framework for nonparametric estimation of Markovian intensities. They survey existing methods such as gradient boosted trees and neural networks and relate them to this setting. Based on real and synthetic data, they find that both gradient boosted trees and neural networks outperform the Cox model in terms of predictive performance as measured by the Brier score. In essence, the framework relies on discretizing time and approximating the intensity with successive Poisson regressions. Using the same idea, [Rytgaard, Eriksson and van der Laan \(2021\)](#) argue that $h(n) = o(n^{-1/4})$ can be achieved for time-independent covariates.

Similarly, [Rytgaard, Gerds and van der Laan \(2022\)](#) mention that $h(n) = o(n^{-1/4})$ can be achieved for estimation of intensities in a multivariate point process with a uniformly bounded number of events, which we place into a general modeling framework below.

D.3. Estimation of λ and Π for counting processes. In Sections 2 and 4 we considered the setup where N was a counting process adapted to a filtration \mathcal{F}_t , which could contain information on baseline covariates and covariate processes that were not necessarily counting processes. In this section we explore how our testing framework can be applied when all stochastic processes of interest are counting processes.

More specifically, let $(N_t^d)_{d \in [p]}$ be a p -dimensional counting process. For $a, b \in [p]$ and $C \subset [p] \setminus \{b\}$ with $a \neq b$ and $a \in C$ we are interested in testing the hypothesis that N^a is conditionally locally independent of N^b given the filtration, \mathcal{F}_t^C , generated by $N^C = (N_t^d)_{d \in C}$.

We can cast this setup in the framework of Section 2 as follows. Naturally, we let $N = N^a$ and $\mathcal{F}_t = \mathcal{F}_t^C$. The auxiliary process X is chosen to be càglàd and predictable with respect to the filtration, \mathcal{F}_t^b , generated by N^b . For example, we could choose $X_t = N_{t-}^b$. But X_t could be any functional of N^b such as $X_t = f(N_{t-}^b)$ for a suitable function f or a linear filter of N^b ,

$$X_t = \int_0^{t-} \kappa(t-s) dN_s^b,$$

where κ is a suitable kernel function, see also Section 3.1. In principle, the process X could also depend on the process N^C , but it is important that the filtration, \mathcal{G}_t , generated by \mathcal{F}_t and X_t is strictly larger than \mathcal{F}_t , i.e., X_t should depend on N^b , in order to get a non-trivial test as explained in Section 2.

In the framework of counting processes, we can approach the estimation of both λ and Π in a unified and general way as follows: Let $(\tau_j, z_j)_{j \geq 1}$ be the marked point process associated with the counting process N^C , i.e., $(\tau_j)_{j \geq 1}$ is a sequence of almost surely strictly increasing event times located at the jumps of N^C , and $(z_j)_{j \geq 1}$ for $z_j \in C$ are the corresponding event types.

Since both λ_t and Π_t are real-valued and \mathcal{F}_{t-}^C -measurable for each fixed $t \geq 0$, they can be represented as measurable functions of $\{(\tau_j, z_j) \mid \tau_j < t, z_j \in C\}$. Hence, we can model both λ and Π using any sequence-to-number model. For the intensity process, [Rytgaard, Gerds and van der Laan \(2022\)](#) propose a sequence of HAL estimators when the total event count is uniformly bounded. As an alternative, [Xiao et al. \(2019\)](#) propose using a recurrent neural network (LSTM). Unless there is a uniform bound on the total number of events, as assumed by [Rytgaard, Gerds and van der Laan \(2022\)](#), there are currently no published results available on the rates of convergence for nonparametric estimation of sequence-to-number functions.

E. Relation to semiparametric survival models. In this section, we relate the LCM to existing work on treatment effects in survival analysis. We resume to the setting of Section 3.2, that is, the case where $N_t = \mathbb{1}(T \geq t)$ is the counting process of a survival time, X is a baseline treatment variable, and where $\mathcal{F}_t = \sigma(N_s, Z; s \leq t)$ for additional baseline covariates Z . Supposing that X is also non-negative, we may consider two different models for the intensity:

$$(12) \quad \lambda_t^\times = \mathbb{1}(T \geq t) \underline{\lambda}(t) \exp(\theta X + \phi(Z)),$$

$$(13) \quad \lambda_t^+ = \mathbb{1}(T \geq t) (\vartheta X + \varphi(t, Z)),$$

where $\theta, \vartheta \in \mathbb{R}$ are treatment parameters of interest, and where $\underline{\lambda}, \phi, \varphi$ deterministic nuisance functions. The model in (12) is known as the partially linear Cox model (PLCM, [Sasieni \(1992\)](#)), and the additive model in (13) was considered by [Dukes et al. \(2019\)](#) among others.

While the parameters θ and ϑ are difficult to compare directly, the hypothesis of conditional local independence corresponds to the hypothesis of zero treatment effect within each of the models, and testing this hypothesis can be done using a score test.

[Sasieni \(1992\)](#) shows that within the PLCM, the efficient score for θ is given by

$$(14) \quad S^\times(\theta; \underline{\lambda}, \phi) = \int_0^1 (X - \alpha^*(t) - h^*(Z)) (dN_t - \mathbb{1}(T \geq t) \underline{\lambda}(t) e^{\theta X + \phi(Z)} dt),$$

where (α^*, h^*) are defined as the minimizers $\mathbb{E}(X - \alpha(T) - h(Z))^2$. Recall that, when X and Z are time-independent, the null hypothesis H_0 of conditional local independence reduces to the conditional independence statement $X \perp\!\!\!\perp T \mid Z$. Consequently, it holds that $\alpha^*(T) = 0$ and $h^*(Z) = \mathbb{E}[X \mid Z] = \Pi_0$ under H_0 . Evaluating the efficient score at $\theta = 0$ under H_0 therefore gives

$$S^\times(0; \underline{\lambda}, \phi) = \int_0^1 (X - \Pi_0) (dN_t - \mathbb{1}(T \geq t) \underline{\lambda}(t) e^{\phi(Z)} dt) = (X - \Pi_0) (1 - \Lambda_T(\underline{\lambda}, \phi)).$$

We see that the empirical version of S^\times is exactly the endpoint of the LCM estimator with additive residual process (cf. Equation 16). This means that our test, the X-LCT, can under the PLCM be interpreted as a score test based on the efficient score.

A similar connection can be made for the additive model. [Dukes et al. \(2019\)](#) show that the efficient score for ϑ is given by

$$S^+(\vartheta; \varphi) = \int_0^1 \left(X - \frac{\mathbb{E}[(\vartheta X + \varphi(t, Z))^{-1} X e^{-\vartheta X t} \mid Z]}{\mathbb{E}[(\vartheta X + \varphi(t, Z))^{-1} e^{-\vartheta X t} \mid Z]} \right) \frac{(dN_t - \lambda_t(\vartheta, \phi) dt)}{(\vartheta X + \varphi(t, Z))}.$$

Plugging in $\vartheta = 0$ and simplifying under H_0 yields

$$S^+(0; \varphi) = \int_0^1 \frac{(X - \mathbb{E}[X | Z])}{\varphi(t, Z)} (dN_t - \mathbb{1}(T \geq t)\varphi(t, Z)dt) = \int_0^1 \frac{(X - \Pi_0)}{\lambda_t} (dN_t - \lambda_t dt).$$

We recognize the empirical version of S^+ as the endpoint of the LCM estimator with the time-constant X replaced by the hazard weighted process X/λ_t .

Other works that consider effect estimation based on orthogonal scores include [Huang \(1999\)](#); [Fang, Ning and Liu \(2017\)](#); [Niu et al. \(2022\)](#); [Zhong, Mueller and Wang \(2022\)](#) for the PLCM and [Hou, Bradic and Xu \(2023\)](#) for the additive model in (13).

We also suspect, as the derivations in Section 3.1 likewise suggest, that the LCM is still an efficient score for certain semiparametric survival models even when the covariates vary with time, but we are not aware of existing results on such a connection.

E. Details on Neyman-orthogonality. In this section, we first show by direct computation that the LCM is Neyman orthogonal with respect to both general residual processes and intensities. We then show that the LCM with an additive residual process can be viewed as a concentrated-out score in the sense of [Newey \(1994\)](#).

E.1. General Neyman orthogonality. The definition of Neyman orthogonality by [Chernozhukov et al. \(2018, Def. 2.1.\)](#) requires that we formally define function spaces for the collections of nuisance parameters. However, to avoid extensive technical specifications (and redundant model assumptions), we prove a simpler – but more general – condition, from which Neyman orthogonality can be derived within specific semiparametric models.

First, we generalize the integral I_t from Definition 2.4 to a function of pairs (x, y) of càglàd functions, given by

$$I_t(x, y) = \int_0^t x_s (dN_s - y_s ds).$$

With this notation, the LCM is given by $\gamma_t = \mathbb{E}[I_t(G, \lambda)]$, where G is a residual process and λ is the \mathcal{F}_t -intensity of N . We assume for simplicity that λ and G are bounded such that the expectation is well-defined.

Now let \tilde{G}_t be an arbitrary bounded \mathcal{G}_t -predictable càglàd process, and let $\tilde{\lambda}_t$ be an arbitrary bounded \mathcal{F}_t -predictable càglàd process. We establish the following orthogonality condition: under H_0 it holds that

$$(15) \quad \partial_r \mathbb{E}[I_t(G + r(\tilde{G} - G), \lambda + r(\tilde{\lambda} - \lambda)) \Big|_{r=0}] = 0.$$

Indeed, observe that

$$\begin{aligned} I_t(G + r(\tilde{G} - G), \lambda + r(\tilde{\lambda} - \lambda)) \\ = (1-r)^2 I_t(G, \lambda) + r(1-r)(I_t(G, \tilde{\lambda}) + I_t(\tilde{G}, \lambda)) + r^2 I_t(\tilde{G}, \tilde{\lambda}), \end{aligned}$$

from which it follows that

$$\partial_r \mathbb{E}[I_t(G + r(\tilde{G} - G), \lambda + r(\tilde{\lambda} - \lambda)) \Big|_{r=0}] = 2 \cdot \mathbb{E}[I_t(G, \lambda)] + \mathbb{E}[I_t(G, \tilde{\lambda})] + \mathbb{E}[I_t(\tilde{G}, \lambda)].$$

The first term is zero under H_0 by Proposition 2.5, and the third term vanishes under H_0 by the same argument. For the second term, we note that

$$\mathbb{E}[I_t(G, \tilde{\lambda})] = \gamma_t + \mathbb{E} \left[\int_0^t G_s (\lambda_s - \tilde{\lambda}_s) ds \right] = \gamma_t + \int_0^t \mathbb{E} \left[\mathbb{E}[G_s | \mathcal{F}_{s-}] (\lambda_s - \tilde{\lambda}_s) \right] ds = \gamma_t,$$

which also vanishes under H_0 . This lets us conclude that (15) holds under H_0 .

F.2. Concentrating-out. To derive the concentrated-out score, we first need to formalize the nuisance parameters and the collection thereof. We consider the case where $\mathcal{F}_t = \mathcal{F}_t^{N,Z}$ for a process $Z = (Z_t)$, and let D_X and D_Z denote the respective sample spaces of the X and Z . We posit the following semiparametric model for the intensity:

$$\lambda_t(\beta, h) = \mathbb{1}(T \geq t) e^{\beta X_t} h(t, Z),$$

where $h: [0, 1] \times D_Z \rightarrow [0, \infty)$ is a function such that $t \mapsto h(t, Z)$ is an \mathcal{F}_t^Z -predictable càglàd process. Denote the collection of such functions by \mathcal{T}_1 . To make the space not dependent on the particular instantiation of the process Z , we could also take it to be the set:

$$\left\{ (h_t)_{t \in [0,1]} \in \int_{[0,1]}^{\oplus} L_2(D_Z|_{[0,t]}, \mathbb{R}) dt \mid t \mapsto h_t((z_s)_{s < t}) \text{ is a bounded nonnegative} \right. \\ \left. \text{càglàd function for all } (z_s)_{0 \leq s \leq 1} \in D_Z \right\},$$

where \int^{\oplus} denotes the direct integral, and where $D_Z|_{[0,t]}$ is the path space D_Z restricted to the domain $[0, t)$.

Recall that the likelihood at $t = 1$ is given by

$$\ell_1(\beta, h) = \int_0^1 \log(\lambda_s(\beta, h)) dN_s - \int_0^1 \lambda_s(\beta, h) ds.$$

We show that concentrating out the nuisance parameter h of $\ell_1(\beta, h)$ yields to the LCM with additive residual process under H_0 .

Let P_0 be fixed distribution satisfying H_0 with ground truth $\beta^0 = 0$ and $h^0 \in \mathcal{T}_1$. Suppose that for each $\beta \in (-\varepsilon, \varepsilon)$ in a neighborhood of zero, there is a function $\varpi_\beta \in \mathcal{T}_1$ such that

$$\varpi_\beta(t, Z) = \mathbb{E}_{P_0}[e^{\beta X_t} \mid T \geq t, \mathcal{F}_{t-}^Z].$$

The first step of concentrating-out is to maximize the expected likelihood over h . We claim that for each fixed β , the function $h_\beta^* := h^0 / \varpi_\beta \in \mathcal{T}_1$ maximizes the objective $u(h) := \mathbb{E}_{P_0}[\ell_1(\beta, h)]$ over $h \in \mathcal{T}_1$.

Since the logarithm is concave and $\lambda(\beta, h)$ is linear in h , it follows that u is a concave objective. Moreover, one can show that h_β^* is the unique critical point of u by equating its Gateaux derivative to zero and invoking the fundamental lemma of the calculus of variations. For simplicity, we settle with verifying (see below) that the Gateaux derivative is zero at h_β^* . From these properties we may conclude that the global maximum of u is attained at h_β^* .

A straightforward differentiation shows that for any $\tilde{h} \in \mathcal{T}_1$,

$$\partial_r \ell_1(\beta, h_\beta^* + r \cdot \tilde{h})|_{r=0} = \int_0^1 \frac{\tilde{h}(s, Z)}{h_\beta^*(s, Z)} dN_s - \int_0^1 \lambda_s(\beta, \tilde{h}) ds.$$

Because $\frac{\tilde{h}(t, Z)}{h_\beta^*(t, Z)}$ is \mathcal{G}_t -predictable and $N_t - \int_0^t \lambda_s(0, h^0) ds$ is a \mathcal{G}_t -martingale, taking expectation under P_0 yields that

$$\begin{aligned} \partial_r u(h_\beta^* + r \cdot \tilde{h})|_{r=0} &= \mathbb{E}[\partial_r \ell_1(\beta, h_\beta^* + r \cdot \tilde{h})|_{r=0}] \\ &= \mathbb{E} \left[\int_0^1 \frac{\tilde{h}(s, Z)}{h_\beta^*(s, Z)} \lambda_s(0, h^0) ds - \int_0^1 \lambda_s(\beta, \tilde{h}) ds \right] \\ &= \mathbb{E} \left[\int_0^1 \mathbb{1}(T \geq s) \left(\varpi_\beta(t, Z) - e^{\beta X_s} \right) \tilde{h}(s, Z) ds \right] \\ &= \int_0^1 \mathbb{E} \left[\mathbb{1}(T \geq s) \left(\varpi_\beta(t, Z) - \mathbb{E}[e^{\beta X_s} \mid T \geq t, \mathcal{F}_{s-}^Z] \right) \tilde{h}(s, Z) \right] ds = 0, \end{aligned}$$

where we have used that $\tilde{h}(s, Z)$ is \mathcal{F}_s -predictable.

Now, by the method of concentrating-out, we are led to consider nuisance functions of the form

$$\eta_h: (-\varepsilon, \varepsilon) \longrightarrow \mathcal{T}_1,$$

$$\eta_h(\beta) = \left((t, z) \mapsto \frac{h(t, z)}{\varpi_\beta(t, z)} \right),$$

for any $h \in \mathcal{T}_1$, which in particular includes $\eta_{h_0}(\beta) = h_\beta^*$. The resulting concentrating-out score is therefore

$$\begin{aligned} \psi(\beta, \eta_h) &= \frac{\partial \ell_1(\beta, \eta_h(\beta))}{\partial \beta} \\ &= \int_0^1 \left(X_s - \frac{\partial_\beta \varpi_\beta(t, Z)}{\varpi_\beta(t, Z)} \right) dN_s - \int_0^1 \left(\frac{X_s}{\varpi_\beta(t, Z)} - \frac{\partial_\beta \varpi_\beta(t, Z)}{\varpi_\beta(t, Z)^2} \right) \boldsymbol{\lambda}(\beta, h) ds \\ &= \int_0^1 \left(X_s - \frac{\mathbb{E}_{P_0}[X_s e^{\beta X_s} | T \geq s, \mathcal{F}_{s-}^Z]}{\mathbb{E}[e^{\beta X_s} | T \geq s, \mathcal{F}_{s-}^Z]} \right) dN_s \\ &\quad - \int_0^1 \left(\frac{X_s}{\mathbb{E}_{P_0}[e^{\beta X_s} | T \geq s, \mathcal{F}_{s-}^Z]} - \frac{\mathbb{E}_{P_0}[X_s e^{\beta X_s} | T \geq s, \mathcal{F}_{s-}^Z]}{(\mathbb{E}_{P_0}[e^{\beta X_s} | T \geq s, \mathcal{F}_{s-}^Z])^2} \right) \boldsymbol{\lambda}(\beta, h) ds. \end{aligned}$$

After plugging in $\beta = 0$ and simplifying we see that

$$\psi(0, \eta_h) = \int_0^1 (X_s - \mathbb{E}[X_s | \mathcal{F}_{s-}]) (dN_s - \boldsymbol{\lambda}(0, h) ds).$$

This shows that the concentrating-out score evaluated at $\beta = 0$ is exactly the score for the endpoint of the LCM estimator.

G. Additional details of simulation study. This section contains additional details, numerical results, and figures related to the simulations of Section 6.

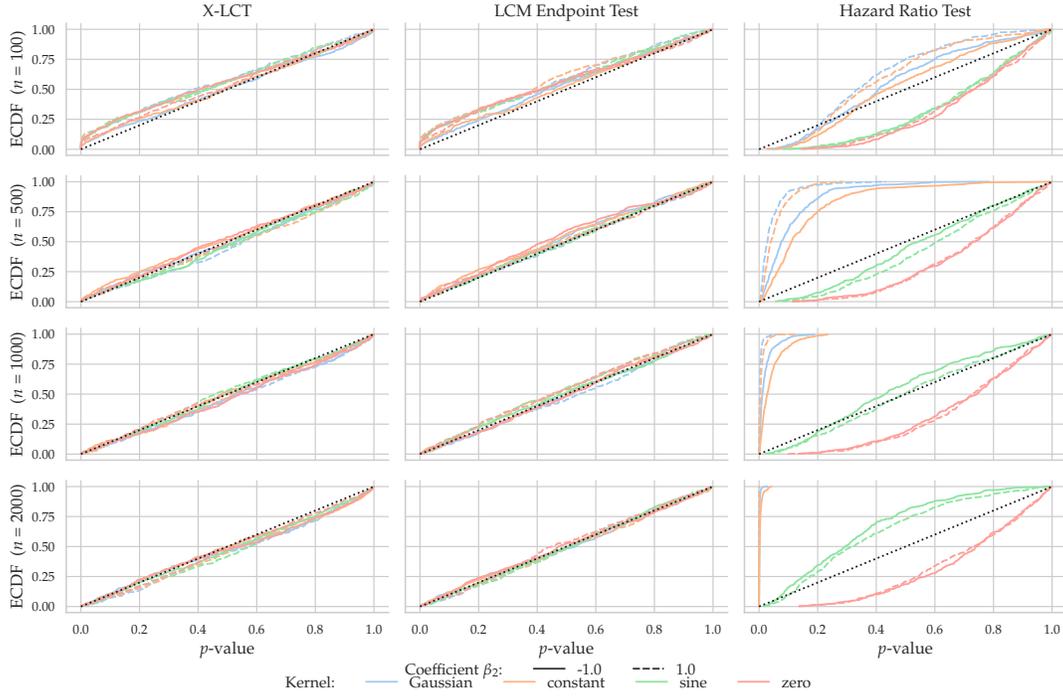


FIG G.1. Empirical distribution functions of p -values for the three different conditional local independence tests considered, simulated under the sampling scheme described in Section 6. The dotted line shows $y = x$ corresponding to a uniform distribution.

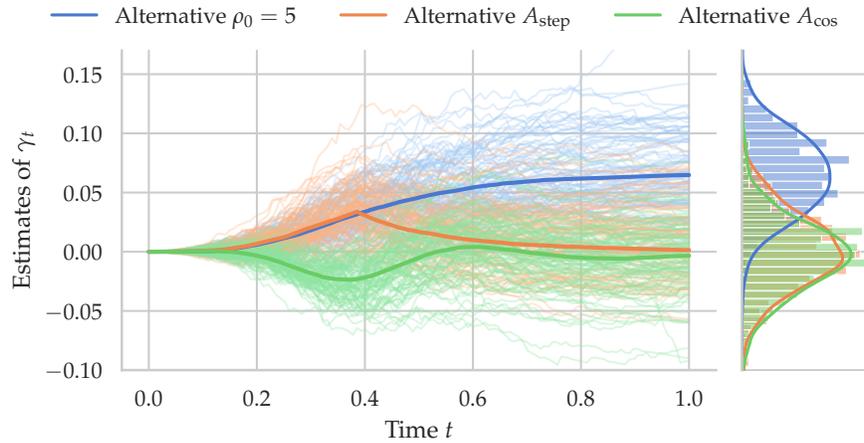


FIG G.2. Sample paths of $\tilde{\gamma}^{K,(500)}$ fitted on data sampled from three different alternatives as described in Section 6.4. Here (X, Y, Z) are sampled from the scheme described in Section 6, with both ρ_X and ρ_Y being the constant kernel and with $\beta = -1$. For each alternative, 100 paths are shown. The empirical mean functions and the endpoint distributions are highlighted and computed based on 500 samples.

G.1. *Implementation of estimators and tests.* For our proof-of-concept implementation we used two simple off-the-shelf estimators.

To estimate λ we used the `BOXHED2.0` estimator from Pakbin et al. (2021), based on the works of Wang et al. (2020) and Lee, Chen and Ishwaran (2021). In essence, the estimator is a gradient boosted forest adapted to the setting of hazard estimation with time-dependent covariates. The maximum depth and number of trees were tuned by 5-fold cross-validation over the same grid as in Pakbin et al. (2021). For computational ease, the hyperparameters were tuned once on the entire dataset instead of tuning them on each fold J_n^k . In principle, this may invalidate the asymptotic properties of $\tilde{\Psi}_n^K$ since it breaks the independence between $\hat{\lambda}^{k,(n)}$ and $(T_j, X_j, Z_j)_{j \in J_n^k}$, but we believe that this dependency is negligible.

To estimate the predictable projection $\Pi_t = \mathbb{E}(X_t | \mathcal{F}_{t-})$, we fitted a series of linear least squares estimators by regressing X_t on $(Z_s)_{s \in \mathbb{T}: s < t}$ for each $t \in \mathbb{T}$. To stabilize the estimation error $g(n)$, we added a small L_2 -penalty with coefficient 0.001 fixed across all experiments for simplicity. Since X_t was sampled from a discretized historical linear model, the error $g(n)$ should in principle converge with a classical $n^{-1/2}$ -rate. The historical linear regression estimator from the `scikit-fda` library was also considered initially, but we found that fitting this model was too computationally expensive for a simulation study with cross-fitting. In principle, in our time-continuous setting, we would like to use a functional estimator of Π that would utilize the regularity along s and t . Initial experiments, however, suggested that the simpler historical regression described above gave similar results as using the `scikit-fda` library, and we went with the less time consuming implementation.

Based on these estimators, the X-LCT was implemented based on Algorithm 2. Following the recommendation by Chernozhukov et al. (2018, Remark 3.1.), we computed the X-LCT with $K = 5$ folds. The associated p -value was computed with the series representation of F_S truncated to the first 1000 terms.

We compared our results for X-LCT with a hazard ratio test in the possibly misspecified marginal Cox model given by (11). This test was computed using the `lifelines` library (Davidson-Pilon, 2021), specifically the `CoxTimeVaryingFitter` model. The model was fitted with an L_2 -penalty with a coefficient set to 0.1 (the default), and as a consequence the hazard ratio test is expected to be conservative.

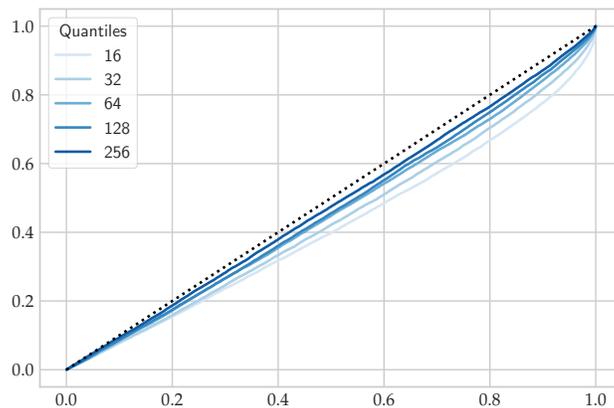


FIG G.3. Empirical distribution functions of $p^{(q)} = 1 - F_S(M^{(q)})$, where $M^{(q)} = (M_t^{(q)})_{t=1, \dots, q}$ is a random walk with Gaussian increments such that $M_q^{(q)}$ has unit variance for each $q \in \{2^\ell : \ell = 4, \dots, 8\}$. Each empirical distribution function is based on $N = 20\,000$ samples.

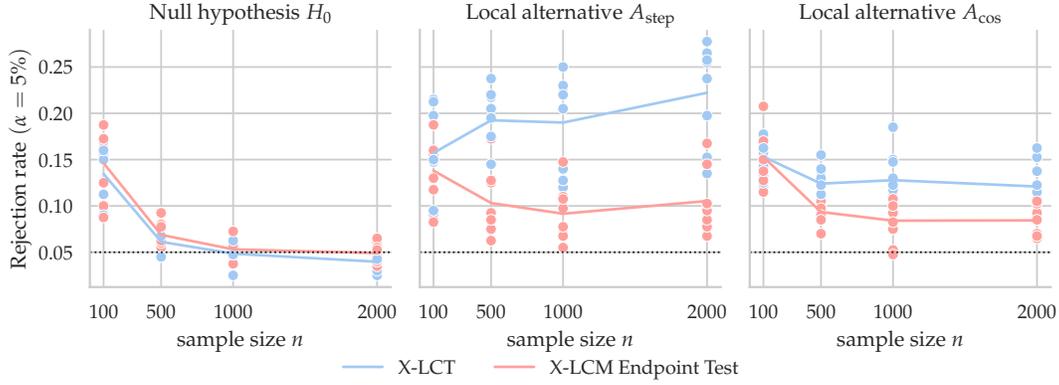


FIG G.4. The plots show the average rejection rate of the double machine learning tests based on the supremum statistic (blue) and the endpoint statistic (red).

G.2. *Comparison with endpoint statistic.* We compare the X-LCT, which is based on the uniform norm of the X-LCM, with its endpoint counterpart. More precisely, we consider the test statistic

$$(\check{\mathcal{V}}_{K,n}(1))^{-\frac{1}{2}} \sqrt{n} \check{\gamma}_1^{K,(n)},$$

which is asymptotically standard normal under H_0 . With the simulation settings in Section 6.4, the X-LCT turns out to be more or less indistinguishable from the corresponding endpoint test. This is because the alternatives considered have corresponding LCMs, which are most extreme towards $t = 1$. Therefore, the supremum and the endpoint behave similarly in these cases.

For this reason we consider local alternatives that result in a non-monotonic LCM. Using the same expression for the intensity (39), but with a time-varying ρ_0 , we consider the alternatives

$$A_{\text{step}}: \rho_0(t) = 5 \cdot \mathbf{1}(t \leq 0.4) - 5 \cdot \mathbf{1}(t > 0.4),$$

$$A_{\text{cos}}: \rho_0(t) = 7 \cdot \cos(4\pi \cdot t).$$

The idea behind the alternative A_{step} is that the LCM should be increasing on $[0, 0.4]$ and decreasing on $(0.4, 1]$. Figure G.2 shows sample paths of $\check{\gamma}^{K,(n)}$ for data simulated under each of the alternatives $\rho_0 = 5$, A_{step} and A_{cos} . The figure illustrates that $t \mapsto |\check{\gamma}_t^{K,(n)}|$ is, indeed, mostly maximal towards $t = 1$ for the alternative $\rho_0 = 5$, but not for the time-varying alternatives A_{step} and A_{cos} .

With the same sampling scheme for (X, Y, Z) as in Section 6.2, we conducted an analogous experiment with 400 runs for each setting. Figure G.4 shows the rejection rates for the two tests.

Under the hypothesis of conditional local independence, the left plot in Figure G.4 shows that the endpoint test behaves similarly to $\check{\Psi}_n^K$ as expected. Both tests have power against the local alternatives, but for A_{step} the power does not seem to stabilize before $n = 2000$. This is different from the previous settings, and can be explained by a slower convergence of the intensity estimator due to the more complex dependency on X . For both of the local alternatives, we observe that $\check{\Psi}_n^K$ is more powerful than the endpoint test, with the difference being largest for A_{step} . In conclusion, these results show that the supremum test dominates the endpoint test in certain situations.

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