Using measure changes to construct stochastic intensity point processes

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Construction of point processes

Consider a filtered probability space $(\Omega, \mathcal{F}, P, \mathcal{F}_t)$ satisfying the usual conditions. Let N be a process with values in the path space of nonexploding point process paths. Let λ be a nonnegative predictable locally bounded process on the same probability space.

We say that N is a point process with intensity λ if it holds that $N_t - \int_0^t \lambda_s \, \mathrm{d}s$ is a local martingale.

Our guiding question: For which processes λ do there exist a point process N with λ as its intensity?

Previous results

The question of existence of nonexploding point process distributions on the canonical space is a well-known problem. In Jacobsen (2005), Theorem 3.1.1, a very general sufficient criterion is given for the existence of nonexplosive point process distributions in terms of conditional event time distributions. In Proposition 4.3.5, a criterion is given in terms of the integrated intensity. The most useful criterion is roughly speaking that if $\lambda_t \leq a(N_{t-})$, where

$$\sum_{n=0}^{\infty} \frac{1}{a(n)} = \infty,$$

then there exists a nonexploding point process distribution with intensity λ. However, this result *depends on the assumption that λ is defined on the canonical path space.*

Our goal

We would like to obtain an existence result of the following type.

Given: A general filtered probability space $(\Omega, \mathcal{F}, P, \mathcal{F}_t)$ satisfying the usual conditions, with a standard Poisson process N and a nonnegative predictable locally bounded process λ.

Result: The existence of a measure Q on $(\Omega, \mathcal{F})$ such that under Q, N is a point process intensity λ.

Prospects and limitations

The consideration of general filtered probability spaces allow us to easily introduce intensities depending on other processes than the point process itself, such as for example $\lambda_t = c + |W_t|$, where W is a Brownian motion.

On the other hand, considering general probability spaces may also mean restricting the range of intensities where results can be obtained. In particular, it is probably utopian to believe that criteria as general as in the canonical case can be easily found.

Current result

Our current main result is as follows.

Theorem. *Let $(\Omega, \mathcal{F}, P, \mathcal{F}_t)$ be a filtered probability space with a standard Poisson process N and a positive, predictable locally bounded process λ. Assume that $\lambda_t \leq \alpha N_{t-} + \beta$. For any $t \geq 0$, there exists a measure Q_t equivalent to P such that under Q_t, N has intensity λ on $[0, t]$.*

Furthermore, we also obtain the classical likelihood directly:

$$\frac{\mathrm{d}Q_t}{\mathrm{d}P} = \exp\left(t - \int_0^t \lambda_s \, \mathrm{d}s + \int_0^t \log \lambda_s \, \mathrm{d}N_s\right).$$

The proof

We outline the proof of the theorem. Letting M be a local martingale with $\Delta M_t > -1$ which is zero at zero, we let $\mathcal{E}(M)$ denote the exponential martingale. Define

$$B_t = \frac{1}{2}[M^c]_t + \sum_{0<s\leq t} (1+\Delta M_s)\log(1+\Delta M_s) - \Delta M_s.$$

Let $\Pi_p^* B$ denote the dual predictable projection. From Lepingle & Mémin (1978), we have the following result: $\mathcal{E}(M)$ is an UI martingale if $\exp(\Pi_p^* B_\infty)$ is integrable. In particular, if $\exp(\Pi_p^* B_t)$ is integrable, $\mathcal{E}(M)_t$ has unit mean and therefore is the density of a measure Q_t on $(\Omega, \mathcal{F})$ with respect to P.

From the Girsanov-Meyer Theorem, we find that the only viable candidate for M in order for N to have intensity λ on $[0, t]$ under Q_t is obtained by putting $M_t = \int_0^t \lambda_s - 1 \, \mathrm{d}(N_s - s)$.

In order to use the result from Lepingle & Mémin (1978), we need the following observation.

Lemma. *Let M be a local martingale with $\Delta M_t > -1$ which is zero at zero, and let $\varepsilon > 0$. If $\mathcal{E}(M^{n\varepsilon} - M^{(n-1)\varepsilon})$ is an UI martingale for all n, then $\mathcal{E}(M)$ is a martingale.*

Calculations show that for our choice of M, we have

$$\Pi_p^* B_t = \int_0^t \lambda_s \log \lambda_s - (\lambda_s - 1) \, \mathrm{d}s.$$

Therefore, in order to obtain that $\mathcal{E}(M)_t$ has unit mean, we merely have to identify sufficient criteria for

$$E \exp\left(\int_{(n-1)\varepsilon}^{n\varepsilon} \lambda_s \log \lambda_s \, \mathrm{d}s\right) < \infty.$$

Some calculations and moment properties of the Poisson distributions show that this is the case if we require $\lambda_t \leq \alpha N_{t-} + \beta$ for some $\alpha, \beta > 0$.

Exogeneous intensity sources

The general criterion that

$$E \exp\left(\int_{(n-1)\varepsilon}^{n\varepsilon} \lambda_s \log \lambda_s \, \mathrm{d}s\right) < \infty$$

can also be used for cases where λ does not only depend on N. For example, by the results established in Gao & Jiang (2009), the criterion covers the case where λ is the absolute value of an Ornstein-Uhlenbeck process.

Further work

Prospects for further results are:

1. The construction of measures Q_∞ where the intensity is changed over all of $[0, \infty)$. Such measures would not retain equivalence with the original measure P. However, on canonical spaces they could possibly be obtained from Q_t by Carathéodory's extension theorem, or a weak convergence argument building on Cherny (2001).
2. Taking λ to satisfy an equation of type $\mathrm{d}\lambda_t = \mu(t, \lambda_{t-})\,\mathrm{d}t + \sigma(t, \lambda_{t-})\,\mathrm{d}W_t + \nu(t, \lambda_{t-})\,\mathrm{d}N_t$, identification of parameters where the measure-change criteria for existence applies. Of particular interest is the case $\nu(t, x) = -(x - c)$. Such "renewal" models could be applied in, say, neuron spike time modeling.

References

References

A. S. Cherny: Families of consistent probability measures, Theory of Probability and its Applications, Vol. 46, No. 1, p. 160-163, 2001.

F. Gao & H. Jiang: Deviation inequalities and moderate deviations for estimators of parameters in an Ornstein-Uhlenbeck process with linear drift, Electronic communications in probability, 2009.

M. Jacobsen: Point Process Theory and Applications: Marked Point and Piecewise Deterministic Processes, Birkhäuser, 2005.

D. Lepingle & J. Mémin: Sur l'intégrabilité uniforme des martingales exponentielles, Zeitschrift für Wahrscheinkeitstheorie und verwandte Gebiete, 42, 175-203, 1978.