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The maximum of a Lévy process reflected at a general barrier

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Abstract

We investigate the reflection of a Lévy process at a deterministic, time-dependent barrier and in particular properties of the global maximum of the reflected Lévy process. Under the assumption of a finite Laplace exponent, $\psi(\theta)$, and the existence of a solution $\theta^* > 0$ to $\psi(\theta) = 0$ we derive conditions in terms of the barrier for almost sure finiteness of the maximum. If the maximum is finite almost surely, we show that the tail of its distribution decays like $K \exp(-\theta^* x)$. The constant K can be completely characterized, and we present several possible representations. Some special cases where the constant can be computed explicitly are treated in greater detail, for instance Brownian motion with a linear or a piecewise linear barrier. In the context of queuing and storage models the barrier has an interpretation as a time-dependent maximal capacity. In risk theory the barrier can be interpreted as a time-dependent strategy for (continuous) dividend pay out.

Keywords: Lévy processes; Kella–Whitt martingale; Reflection; Risk theory; Nonlinear barriers; Storage models; Queuing theory

1. Introduction

The reflection of a one-dimensional Lévy process at a deterministic, time-dependent barrier presents itself as a natural generalization of the reflection at 0, which in itself is a central object in the study of storage and queuing models. The Lévy process reflected at 0 provides a model for the stored volume or workload in a system — see Chapter 4 in [1] or [2] for more details. For Brownian motion we can also interpret the reflected process at zero as the path of a one-dimensional

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Brownian particle, whose movements are restricted by a fixed barrier at 0. This process is sometimes referred to as regulated Brownian motion and for a thorough treatment and examples of its use we refer to [3]. The purpose of this paper is to study the reflection of a Lévy process at a deterministic, time-dependent barrier given in terms of a function $g : [0, \infty) \to (-\infty, 0]$. As for the reflection at 0 the reflected process at g can be given an explicit representation as $W_t = X_t + L_t$ where $(X_t)_{t\geq 0}$ is the Lévy process and $(L_t)_{t\geq 0}$ is a local time[1] for $(W_t)_{t\geq 0}$ at the barrier g. This representation is used for instance in [4], where the barrier is itself the sample path of a stochastic process (a Poisson age process). In [5] the reflection of one-dimensional Brownian motion at the path of another Brownian motion is considered, see also Section 3 in [6] for general issues regarding the reflection of one-dimensional Brownian motion. In dimensions >1 the mere existence of the reflected process – even for Brownian motion in a time-*independent* domain – is a non-trivial matter. Some smoothness assumptions on the domain are required. For C^3 time-*dependent* domains, the reflection of n-dimensional Brownian motion is treated in [6].

In this paper we restrict our attention to the one-dimensional case, where we investigate the tail behavior of the maximum of the reflected process assuming that there is a solution $\theta^* > 0$ to the equation $\psi(\theta) = 0$ where $\psi(\theta)$ denotes the Laplace exponent. Thus we impose a light-tailed assumption on the positive jumps of the Lévy process. In [7] the similar setup in discrete time was considered, and results on the tail behavior of the global maximum of a random walk reflected at a general, deterministic barrier were derived. The main motivation behind [7] was a problem from structural biology, which is treated thoroughly in [8]. One – purely mathematical – motivation for considering the reflection of a Lévy process at a deterministic, nonlinear barrier is that it provides an example of a time *inhomogeneous* process where we can give quite precise results on the (asymptotic) distribution of its maximum. But several interpretations are also possible. In Section 4 the interpretation of the reflected Lévy process in the context of storage models is discussed, and we show that the barrier can be interpreted as a strategy for future expansion of the storage capacity. A corresponding interpretation for the Cramér–Lundberg risk model is also possible, where the barrier represents the future threshold for (continuous) dividend pay out.

2. Setup

In this paper we will throughout let $(X_t)_{t\geq 0}$ be a Lévy process with $X_0 = 0$ and $g : [0, \infty) \to (-\infty, 0]$ a càdlàg function. We assume that $(X_t)_{t\geq 0}$ is the canonical coordinate projection process defined on the space $D([0, \infty), \mathbb{R})$ of càdlàg functions on $[0, \infty)$ with values in $\mathbb{R}$, which is equipped with the natural filtration

$$\mathcal{F}_t = \sigma\{X_s, 0 \leq s \leq t\}$$

and the probability measure $\mathbb{P}$. We will assume that the Laplace exponent $\psi(\theta) = \log \mathbb{E}(\exp(\theta X_1))$ for the Lévy process is finite in an interval $[0, \beta]$ for some $\beta > 0$, and that there exists a solution $\theta^* \in (0, \beta)$ to the equation $\psi(\theta) = 0$. Since the Laplace exponent is a convex function, such a solution is necessarily unique. This is the Cramér condition, which implies that $X_t \to -\infty$ $\mathbb{P}$-a.s. for $t \to \infty$. We will also assume that $g(t) \to -\infty$ for $t \to \infty$. For the general construction of the reflected process, the assumptions that g takes negative values and tends to $-\infty$ are not essential, but the results we present are only of interest under assumptions that force $g(t) \to -\infty$ for $t \to \infty$ and thus g to be negative eventually.

[1] We use *local time* simply to denote an increasing process that in this case increases only when $W_t = g(t)$.

Define the process

$$L_t = \sup_{0 \le s \le t} \{\max\{g(s) - X_s, 0\}\},$$

together with

$$W_t = X_t + L_t.$$

The process $(W_t)_{t\geq 0}$ is always above the *barrier* given by g, that is, $W_t \geq g(t)$ for all $t \geq 0$. We call $(W_t)_{t\geq 0}$ the reflection of $(X_t)_{t\geq 0}$ at the barrier given by g. We also note that $(L_t)_{t\geq 0}$ is an increasing, càdlàg stochastic process, which increases only when $W_t = g(t)$, that is to say

$$\int_0^\infty 1(W_t > g(t))\,\mathrm{d}L_t = 0.$$

We may call $(L_t)_{t\geq 0}$ a *local time* for the process $(W_t)_{t\geq 0}$ at the barrier g.

We also recall that the *exponential martingale* $(\exp(\theta^* X_t))_{t\geq 0}$ defines a new probability measure — the exponentially tilted measure or Esscher transformed measure $\mathbb{P}^*$ on $D([0,\infty),\mathbb{R})$. See Chapter XIII in [2] for details. The characterization of the measure $\mathbb{P}^*$ is that the restriction of $\mathbb{P}^*$ to $\mathcal{F}_t$ for all $t \geq 0$ has the Radon–Nikodym derivative $(\exp(\theta^* X_s))_{0\leq s\leq t}$ w.r.t. the restriction of $\mathbb{P}$ to $\mathcal{F}_t$. According to Theorem XIII.3.4 in [2] the process $(X_t)_{t\geq 0}$ is also a Lévy process under $\mathbb{P}^*$. We may observe that X_t has finite expectation under $\mathbb{P}^*$, that $\mathbb{E}^*(X_t) = t\psi'(\theta^*) > 0$ and thus that $X_t \to \infty$ $\mathbb{P}^*$-a.s. for $t \to \infty$. Consequently, as g is negative,

$$L_\infty := \lim_{t\to\infty} L_t = \sup_{t\geq 0} L_t < \infty, \ \ \mathbb{P}^*\text{-a.s.}$$

3. Results

The first theorem gives a criterion for finiteness of the global maximum of the reflected process in terms of the distribution of L_∞ under $\mathbb{P}^*$ and provides a Cramér–Lundberg type of inequality.

Theorem 1. *With $\mathcal{M} = \sup_{t\geq 0} W_t$ the global maximum of the reflected process $(W_t)_{t\geq 0}$ it holds in the setup as described above that*

$$\mathbb{P}(\mathcal{M} > u) \leq \mathrm{e}^{-\theta^* u}\mathbb{E}^*\left(\mathrm{e}^{\theta^* L_\infty}\right),$$

and $\mathbb{P}(\mathcal{M} < \infty) = 1$ if and only if

$$\mathbb{E}^*\left(\mathrm{e}^{\theta^* L_\infty}\right) < \infty.$$

With $\tau(u) = \inf\{t > 0 \mid X_t > u\}$ the overshoot $X_{\tau(u)} - u$ has under $\mathbb{P}^*$ a limit distribution for $u \to \infty$ if and only if $(X_t)_{t\geq 0}$ is *not* a compound Poisson process with a lattice jump distribution. If we exclude the lattice case this result follows from Theorem 8 in [9] since under $\mathbb{P}^*$ we have that X_1 has finite, positive mean, hence

$$X_{\tau(u)} - u \xrightarrow{\mathcal{D}} B_\infty \tag{1}$$

where B_∞ is a positive, random variable.

Theorem 2. *With the setup as for Theorem 1 and with the non-lattice assumption as above then if* $\mathbb{E}^*(\exp(\theta^* L_\infty)) < \infty$ *it holds that*

$$\mathbb{P}(\mathcal{M} > u) \sim \mathrm{e}^{-\theta^* u}\mathbb{E}^*\left(\mathrm{e}^{-\theta^* B_\infty}\right)\mathbb{E}^*\left(\mathrm{e}^{\theta^* L_\infty}\right) \tag{2}$$

for $u \to \infty$.

Theorem 2 states that the probability $\mathbb{P}(\mathcal{M} > u)$ decays exponentially with a constant of proportionality that is composed of two factors. The factor $\mathbb{E}^*(\exp(-\theta^* B_\infty))$ is well-known from the Cramér–Lundberg result, see Theorem 7.6 in [1]. We may note that for a spectrally negative Lévy process the overshoot above level u is always 0, hence the factor is in this case always equal to 1. We will not pursue any further discussions of this factor.

The second factor, $\mathbb{E}^*(\exp(\theta^* L_\infty))$, completely characterizes the influence of the barrier on the asymptotic result above, and we present several alternative representations below. The formula in Theorem 3 provides some additional insights into the role of g. Imposing mild regularity assumptions on g we provide a different representation in Corollary 5 and a quite explicit criterion for verifying whether the factor is finite, and thus whether $\mathbb{P}(\mathcal{M} < \infty) = 1$

For the reflection at the zero barrier, the reflected process will eventually overshoot any level. In contrast Theorem 2 above deals with barriers that decay sufficiently fast to $-\infty$. Theorem 1 in [10] shows that for the reflection at zero the number of overshoots in a large interval over a suitably chosen threshold follows asymptotically a homogeneous Poisson process. It would be of interest to study what happens if the barrier is chosen so that it decays to $-\infty$ but sufficiently slowly so that the reflected process still overshoots any level. It may be conjectured that for certain such barriers one would obtain an inhomogeneous Poisson process limit for the number of overshoots of a high level in a large interval.

To state the results we introduce the jumps

$$\Delta L_t = L_t - L_{t-}, \quad t \geq 0$$

of the increasing stochastic process $(L_t)_{t\geq 0}$ and we denote by

$$L_t^c = L_t - \sum_{0\leq s\leq t} \Delta L_s,$$

the remaining, increasing and continuous process. In terms of these definitions we introduce the increasing process

$$\tilde{L}_t = \theta^* L_t^c + \sum_{0\leq s\leq t}\left[1 - \mathrm{e}^{-\theta^* \Delta L_s}\right].$$

The processes $(L_t^c)_{t\geq 0}$ and $(\tilde{L}_t)_{t\geq 0}$ (and $(L_t)_{t\geq 0}$ for that matter) are increasing and stochastic integration w.r.t. these processes can therefore be defined pathwise as ordinary Lebesgue–Stieltjes integration.

Theorem 3. *With the above notation*

$$\mathbb{E}^*\left(\mathrm{e}^{\theta^* L_\infty}\right) = 1 + \theta^*\mathbb{E}\left(\int_0^\infty \mathrm{e}^{\theta^* g(t)}\,\mathrm{d}L_t^c\right) + \mathbb{E}\left(\sum_{t\geq 0}\mathrm{e}^{\theta^* g(t)}(1 - \mathrm{e}^{-\theta^*\Delta L_t})\right) \tag{3}$$

$$= 1 + \mathbb{E}\left(\int_0^\infty \mathrm{e}^{\theta^* g(t)}\,\mathrm{d}\tilde{L}_t\right). \tag{4}$$

Remark 4. This formula can be seen as a consequence of the Kella–Whitt martingale, see [11] or Section IX.3 in [2]. In Section 5 we provide two proofs of Theorem 3. One utilizes stochastic analysis and clarifies the relation to the Kella–Whitt martingale and stochastic exponentials and logarithms. The other is much more elementary. Here we point out the immediate relation to stochastic logarithms. Indeed, with $\hat{L} = \mathcal{L}\text{og}(\exp(\theta^* L_t))$ – the stochastic logarithm of $(\exp(\theta^* L_t))_{t\geq 0}$ in the sense of Section II.8 in [12] – we have by II.8.11 in [12] that

$$\begin{aligned}\hat{L}_t &= \mathcal{L}\text{og}(e^{\theta^* L_t}) = \theta^* L_t + \sum_{0\leq s\leq t} [e^{\theta^* \Delta L_s} - 1 - \theta^* \Delta L_s] \\ &= \theta^* L_t^c + \sum_{0\leq s\leq t} [e^{\theta^* \Delta L_s} - 1] = \exp(\theta^* \Delta L_t) \bullet \tilde{L}_t.\end{aligned}$$

On the other hand, Theorem II.8.3 in [12] gives a representation of $\hat{L}_t$ as a stochastic integral,

$$\hat{L}_t = \exp(-\theta^* L_{t-}) \bullet \exp(\theta^* L_t).$$

Thus

$$\exp(\theta^* L_{t-}) \bullet \hat{L}_t = \exp(\theta^* L_t) - 1,$$

which for instance implies that $\exp(\theta^* L_t) = 1 + \exp(\theta^* L_t) \bullet \tilde{L}_t$. For more details we refer to Section II.8 in [12].

The result in Theorem 3 does not impose conditions on the regularity of the barrier g. If we require more regularity of g it is possible to further rewrite the stochastic integral representation using integration by parts. At the same time we obtain a sufficient criterion for finiteness of $\mathbb{E}^* (\exp(\theta^* L_\infty))$. If g is a function of locally bounded variation so is $F_g(t) = 1 - \exp(\theta^* g(t))$. Moreover, $F_g(t)$ is the measure of $[0, t]$ for a (in general signed) Lebesgue–Stieltjes measure on $[0, \infty)$, and since F_g is a function of locally bounded variation we can write $F_g = F_g^+ - F_g^-$ where both F_g^+ and F_g^- are positive, increasing functions — in correspondence with the Jordan–Hahn decomposition of the measure given by F_g.

Corollary 5. *Assume that g has locally bounded variation. If either g is non-increasing or if $\mathbb{E}|X_1| < \infty$ a sufficient condition for $\mathbb{E}^* (\exp(\theta^* L_\infty))$ to be finite is that*

$$\int_0^\infty t F_g^+(\mathrm{d}t) < \infty. \tag{5}$$

In that case

$$\mathbb{E}^* \left(e^{\theta^* L_\infty}\right) = 1 + \int_0^\infty \mathbb{E}(\tilde{L}_{t-}) F_g(\mathrm{d}t). \tag{6}$$

Additional restrictions on g allow us to drop the left limit in the integrand in (6) and replace $\mathbb{E}(\tilde{L}_{t-})$ with $\mathbb{E}(\tilde{L}_t)$. One trivial example is when g is continuous. But the integrand can also be replaced by $\mathbb{E}(\tilde{L}_t)$ if g is non-increasing. Indeed, when g is non-increasing a jump of $\tilde{L}_t$ can only be due to a jump in the Lévy process, and because F_g has only a countable number of discontinuities (which are all positive jumps when g is non-increasing), the probability that the Lévy process has a jump at any of these discontinuities is 0. In addition, we state in the theorem that for g non-increasing we do not need the first moment assumption on X_1. Of course, in the setup of this paper, X_1 has a light positive tail, thus X_1^+ has moments of all orders, but we impose no general restrictions on the negative part. That is to say, $(X_t)_{t\geq 0}$ can have negative jumps without finite first moment. In that case a (continuous) g with valleys getting rapidly deeper for $t \to \infty$ followed by sharp increases – but still with (5) fulfilled – can have the effect of "reversing" large negative jumps of $(X_t)_{t\geq 0}$ into (almost) equally large positive increases of L_t^c. Consequently, without the first moment assumption on X_1, we risk that $\mathbb{E}(\tilde{L}_{t-})$ increases faster than a linear function in t, and (5) will not be sufficient to assure integrability of the r.h.s. in (6). Such madness cannot arise if g is monotone, say.

Remark 6. If g is non-increasing then F_g is non-decreasing and $F_g^+ = F_g$. Due to the general assumption that $g(t) \in (-\infty, 0]$ and $g(t) \to -\infty$ for $t \to \infty$ we have that $F_g(t) \in [0, 1]$ and $F_g(t) \to 1$ for $t \to \infty$. Thus defining $F_g(t) = 0$ for $t < 0$, F_g becomes a distribution function for a probability measure concentrated on $[0, \infty)$ and (5) expresses that this measure should have finite first moment. We know, by integration by parts (again), that

$$\int_0^\infty t F_g(\mathrm{d}t) = \int_0^\infty \mathrm{e}^{\theta^* g(t)}\,\mathrm{d}t,$$

hence (5) is equivalent to

$$\int_0^\infty \mathrm{e}^{\theta^* g(t)}\,\mathrm{d}t < \infty \tag{7}$$

when g is non-increasing. Moreover, as argued above (as a consequence of Lemma 10) for g non-increasing we do not need finiteness of $\mathbb{E}|X_1|$ to get integrability of $\mathbb{E}(\tilde{L}_t)$. Hence for g non-increasing (7) is a sufficient criterion for finiteness of $\mathbb{E}^*(\exp(\theta^* L_\infty))$, in which case

$$\mathbb{E}^*\left(\mathrm{e}^{\theta^* L_\infty}\right) = 1 + \mathbb{E}(\tilde{L}_\tau). \tag{8}$$

where τ is a random variable, independent of the Lévy process, with distribution given by F_g. This formula immediately suggests how to obtain an unbiased estimate of $\mathbb{E}^*(\exp(\theta^* L_\infty))$ by simulating i.i.d. F_g-distributed stopping times, and then simulating, independently, $\tilde{L}_t$-processes up to the stopping times.

Remark 7. If g is not monotone we may not be able to verify (5) directly. If $\bar{g}(t) \geq g(t)$ for all $t \geq 0$ then

$$L_t = \sup_{0\leq s\leq t}\{\max\{g(s) - X_s, 0\}\} \leq \sup_{0\leq s\leq t}\{\max\{\bar{g}(s) - X_s, 0\}\} = \bar{L}_t,$$

and in particular $\mathbb{E}^*(\exp(\theta^* L_\infty)) \leq \mathbb{E}^*(\exp(\theta^* \bar{L}_\infty))$. Hence if $\bar{g}$ is non-increasing and fulfills (7), which is directly verifiable in terms of g, then $\mathbb{E}^*(\exp(\theta^* L_\infty)) < \infty$. Note that one can always take $\bar{g}(t) = \frac{1}{\theta^*}\log(1 - F_g^+(t))$ as a monotone function that dominates g. Then the circle is complete and we end up with condition (5) again.

4. Examples

We divide this section of examples into four subsections. Section 4.1 deals with the special case of a linear barrier $g(t) = -\alpha t$ for $\alpha > 0$, where we can obtain some explicit representations of $\mathbb{E}^*(\exp(\theta^* L_\infty))$ — especially for spectrally positive Lévy processes. Section 4.2 deals with a generalization to the piecewise linear barrier $g(t) = \min\{0, -\alpha(t - t_0)\}$ for $\alpha, t_0 > 0$ for spectrally positive processes. In Section 4.3 we specialize the Lévy process to be a Brownian motion with drift and give completely explicit results for the linear and piecewise linear barrier. Section 4.4 focuses on interpretations in the context of storage models, and we explore the possibilities of computing the factor $\mathbb{E}^*(\exp(\theta^* L_\infty))$ analytically using either direct arguments or some of the alternative representations presented above.

4.1. Linear barrier

If the barrier $g(t) = -\alpha t$ for $\alpha > 0$, the process $-\alpha t - X_t$ is again a Lévy process, and L_∞ is thus the global maximum of a Lévy process. It follows easily, using (7), that $\mathbb{E}^*(\exp(\theta^* L_\infty)) < \infty$. Using (8) we note that

$$\mathbb{E}^*\left(\mathrm{e}^{\theta^* L_\infty}\right) = 1 + \mathbb{E}(\tilde{L}_\tau)$$

where τ follows an exponential distribution with parameter $\theta^*\alpha$.

If $(X_t)_{t\geq 0}$ is spectrally *positive*, the process $(-\alpha t - X_t)_{t\geq 0}$ is spectrally negative. Corollary VII.2(ii) in [13] implies that L_∞ under $\mathbb{P}^*$ follows an exponential distribution with parameter $\theta_0 > \theta^*$, which is the unique solution bigger than θ^* to the equation $\psi(\theta^* - \theta) - \alpha\theta = 0$. Hence

$$\mathbb{E}^*\left(\mathrm{e}^{\theta^* L_\infty}\right) = \frac{\theta_0}{\theta_0 - \theta^*}.$$

An interesting digression at this point is to interchange the roles of $\mathbb{P}$ and the Esscher transformed measure $\mathbb{P}^*$ and use (8) to give a general expression for the Laplace transform of the maximum of a Lévy process. Let $(X_t)_{t\geq 0}$ be a Lévy process under $\mathbb{Q}$ with ψ^X the Laplace exponent. We assume ψ^X to be finite on an interval, and we assume the existence of $\theta_0 > 0$ with $\psi^X(\theta_0) = 0$. Taking $\theta \in (0, \theta_0)$ we let $\mathbb{Q}^\theta$ denote the Esscher transform of $\mathbb{Q}$. The Radon–Nikodym derivative of $\mathbb{Q}^\theta$ w.r.t. $\mathbb{Q}$ when the measures are restricted to $\mathcal{F}_t$ equals $\exp(\theta X_t - t\psi^X(\theta))$. If we take $\alpha = -\psi^X(\theta)/\theta > 0$ and define the process $Y_t = -\alpha t - X_t$ then $X_t = -\alpha t - Y_t$, and the Lévy process $(Y_t)_{t\geq 0}$ has, under $\mathbb{Q}^\theta$, the Laplace exponent

$$\psi^Y(\theta') = \psi^X(\theta - \theta') - \psi^X(\theta) - \theta'\alpha = \psi^X(\theta - \theta') + \alpha(\theta - \theta').$$

We observe that $\theta > 0$ is the unique, positive solution to $\psi^Y(\theta) = 0$. Taking $\mathbb{P} = \mathbb{Q}^\theta$ its Esscher transform $\mathbb{P}^*$, using the parameter $\theta^* = \theta$, defined in terms of the Lévy process $(Y_t)_{t\geq 0}$, equals $\mathbb{Q}$. Moreover, $L_t = \sup_{0\leq s\leq t}\{-\alpha t - Y_t\} = \sup_{0\leq s\leq t} X_s$, $L_\infty = \sup_{t\geq 0} X_t$ and using (8) we conclude that

$$\mathbb{E}^{\mathbb{Q}}\left(\mathrm{e}^{\theta L_\infty}\right) = 1 + \mathbb{E}^{\mathbb{Q}^\theta}(\tilde{L}_\tau),$$

where τ follows an exponential distribution with parameter $-\psi^Y(\theta)$ and

$$\tilde{L}_t = \theta L_t^c + \sum_{0\leq s\leq t}\left[1 - \mathrm{e}^{-\theta\Delta L_s}\right].$$

Since $\alpha > 0$ for $\theta \in (0, \theta_0)$ we know from (7) that the Laplace transform is finite.

For the special case of a spectrally negative process $(X_t)_{t\geq 0}$, the process $(\tilde{L}_t)_{t\geq 0}$ equals $(\theta L_t)_{t\geq 0}$. It is well established, see Corollary VII.2(i) in [13] or page 213 in [1], that L_τ under $\mathbb{Q}^\theta$ follows an exponential distribution with parameter $\theta_0 - \theta$. Thus

$$1 + \mathbb{E}^{\mathbb{Q}^\theta}(\tilde{L}_\tau) = 1 + \theta\mathbb{E}^{\mathbb{Q}^\theta}(L_\tau) = 1 + \frac{\theta}{\theta_0 - \theta} = \frac{\theta_0}{\theta_0 - \theta}.$$

This is of course in concordance with the fact used above, Corollary VII.2(ii) in [13], that L_∞ under $\mathbb{Q}$ follows an exponential distribution with parameter θ_0.

4.2. Piecewise linear barrier for spectrally positive processes

Consider here the barrier $g(t) = \min\{0, -\alpha(t - t_0)\}$ for $\alpha, t_0 > 0$ and assume that $(X_t)_{t\geq 0}$ is spectrally positive. For $t \geq 0$ we find that

$$\begin{aligned} L_{t+t_0} &= \max\{L_{t_0}, \sup_{0\leq s\leq t}\{g(s+t_0) - X_{s+t_0}\}\} \\ &= \max\left\{L_{t_0}, \overbrace{\sup_{0\leq s\leq t}\{-\alpha s - (X_{s+t_0} - X_{t_0})\}}^{\check{L}_t} - X_{t_0}\right\} \\ &= \max\{L_{t_0}, \check{L}_t - X_{t_0}\}, \end{aligned}$$

and the process $(\check{L}_t)_{t\geq 0}$ is independent of $(L_{t_0}, -X_{t_0})$. Moreover, the independent stopping time τ in (8) equals $t_0 + \sigma$ where σ is exponentially distributed with parameter $\theta^*\alpha$. Thus

$$L_\tau = \max\{L_{t_0}, \check{L}_\sigma - X_{t_0}\} = L_{t_0} + (\check{L}_\sigma - X_{t_0} - L_{t_0})1(\check{L}_\sigma - X_{t_0} > L_{t_0})$$

where $\check{L}_\sigma$ is independent of $(L_{t_0}, -X_{t_0})$ and $L_{t_0} = \sup_{0\leq s\leq t_0}\{-X_s\}$. We let ν denote the distribution of $(L_{t_0}, -X_{t_0})$ on $[0,\infty) \times \mathbb{R}$. Since we assume that $(X_t)_{t\geq 0}$ is spectrally positive we argued in Section 4.1 that $\check{L}_\sigma$ follows an exponential distribution with parameter $\eta > 0$. In the notation of Section 4.1, $\eta = \theta_0 - \theta^*$. Using Tonelli we find

$$\begin{aligned} &\mathbb{E}\left((\check{L}_\sigma - X_{t_0} - L_{t_0})1(L_{t_0} < \hat{L}_\sigma - X_{t_0})\right) \\ &\quad = \frac{1}{\eta}\int_0^\infty \int (z + y - x)1(x < z + y)\nu(\mathrm{d}x, \mathrm{d}y)\mathrm{e}^{-\eta z}\,\mathrm{d}z \\ &\quad = \int \mathrm{e}^{\eta(y-x)}\left[\frac{1}{\eta}\int_{x-y}^\infty (z + y - x)\mathrm{e}^{-\eta(z+y-x)}\,\mathrm{d}z\right]\nu(\mathrm{d}x, \mathrm{d}y) \\ &\quad = \frac{1}{\eta}\int \mathrm{e}^{\eta(y-x)}\nu(\mathrm{d}x, \mathrm{d}y). \end{aligned}$$

This gives that

$$\begin{aligned} \mathbb{E}(L_\tau) &= \int x\nu(\mathrm{d}x, \mathrm{d}y) + \frac{1}{\eta}\int \mathrm{e}^{\eta(y-x)}\nu(\mathrm{d}x, \mathrm{d}y) \\ &= \mathbb{E}(L_{t_0}) + \frac{1}{\eta}\mathbb{E}\left(\mathrm{e}^{-\eta(X_{t_0}+L_{t_0})}\right). \end{aligned}$$

We may note that $X_{t_0} + L_{t_0}$ is the reflection of $(X_t)_{t\geq 0}$ at the zero barrier at time t_0 and by duality

$$X_{t_0} + L_{t_0} \overset{\mathcal{D}}{=} M_{t_0} := \sup_{0\leq s\leq t_0} X_s,$$

see for instance Lemma 3.5 in [1]. Since $(X_t)_{t\geq 0}$ is spectrally positive we also have that $\mathbb{E}(\tilde{L}_\tau) = \theta^*\mathbb{E}(L_\tau)$, hence by (8)

$$\mathbb{E}^*\left(\mathrm{e}^{\theta^* L_\infty}\right) = 1 + \theta^*\mathbb{E}(L_\tau) = 1 + \theta^*\mathbb{E}(L_{t_0}) + \frac{\theta^*}{\eta}\mathbb{E}\left(\mathrm{e}^{-\eta M_{t_0}}\right).$$

4.3. Brownian motion with drift

If $(X_t)_{t\geq 0}$ is a Brownian motion with drift $-\xi < 0$ (and unit variance), then $\psi(\theta) = \log \mathbb{E}(\exp(\theta X_1)) = \frac{\theta^2}{2} - \theta\xi$ and $\theta^* = 2\xi$. Under $\mathbb{P}^*$ the process $(X_t)_{t\geq 0}$ is still a Brownian motion but with drift $\xi > 0$. Since the sample path is continuous (more precisely, upwards skip free), we have that the overshoots $X_{\tau(u)} - u = 0$ for any level u and consequently for $u \to \infty$ we have for that the limit B_∞ equals 0, hence $\mathbb{E}^*(\exp(-\theta^* B_\infty)) = 1$.

If $g(t) = -\alpha t$ is linear with $\alpha > 0$ then according to Section 4.1, $\mathbb{E}^*(\exp(\theta^* L_\infty)) = \frac{\theta_0}{\theta_0 - \theta^*}$ where $\theta_0 > 2\xi$ solves

$$\frac{(2\xi - \theta)^2}{2} - (2\xi - \theta)\xi - \alpha\theta = \frac{\theta^2}{2} - (\alpha + \xi)\theta = 0.$$

Hence $\theta_0 = 2(\alpha + \xi)$ and

$$\mathbb{E}^*\left(e^{\theta^* L_\infty}\right) = \frac{2(\alpha + \xi)}{2(\alpha + \xi) - 2\xi} = \frac{\alpha + \xi}{\alpha} = 1 + \frac{\xi}{\alpha}.$$

If $g(t) = \min\{0, -\alpha(t - t_0)\}$ for $\alpha, t_0 > 0$ we need, according to Section 4.2, to find the Laplace transform of the maximum of Brownian motion over $[0, t_0]$. Note first, that in the notation of Section 4.2 we have here $\eta = \theta_0 - \theta^* = 2\alpha$.

By [14], page 250, formula 2.1.1.3 (see also [15], Section 6) we find that

$$\mathbb{E}\left(e^{-2\alpha M_{t_0}}\right) = \frac{2\alpha + \xi}{\alpha + \xi} e^{2\alpha t_0(\alpha + \xi)} \Phi(-(2\alpha + \xi)\sqrt{t_0}) + \frac{\xi}{\alpha + \xi} \Phi(\xi\sqrt{t_0})$$

where Φ is the distribution function for the standard normal distribution. Moreover, since L_{t_0} is the supremum over $[0, t_0]$ of a Brownian motion with drift ξ (in contrast to M_{t_0} where the drift is $-\xi$), we have the same formula for the Laplace transform of L_{t_0} just with ξ replaced by $-\xi$. Differentiation yields that

$$\mathbb{E}(L_{t_0}) = \frac{1 + \xi^2 t_0}{\xi} \Phi\left(\xi\sqrt{t_0}\right) + \sqrt{\frac{t_0}{2\pi}} e^{-\xi^2 t_0/2} - \frac{1}{2\xi}.$$

Collecting the results and rearranging we obtain

$$\begin{aligned}
\mathbb{E}^*(e^{\theta^* L_\infty}) &= 1 + 2\xi\mathbb{E}(L_{t_0}) + \frac{\xi}{\alpha}\mathbb{E}\left(e^{-2\alpha M_{t_0}}\right) \\
&= \left(2 + 2\xi^2 t_0 + \frac{\xi^2}{\alpha(\alpha + \xi)}\right) \Phi\left(\xi\sqrt{t_0}\right) + 2\xi\sqrt{\frac{t_0}{2\pi}} e^{-\xi^2 t_0/2} \\
&\quad + \frac{\xi(2\alpha + \xi)}{\alpha(\alpha + \xi)} e^{2\alpha t_0(\alpha + \xi)} \Phi(-(2\alpha + \xi)\sqrt{t_0}).
\end{aligned}$$

For details on the distribution of the reflected Brownian motion at 0 and the running maximum of Brownian motion we refer to [3], and for details on the computation of the Laplace transform and the first moment(s) we refer to [15], Section 6 and Theorem 1.1. Fig. 1 shows the behavior of $\mathbb{E}^*(\exp(\theta^* L_\infty))$ for $\xi = 1$ as a function of α and t_0. As we can also see from the expression, the dominating behavior for $t_0 \simeq 0$ is as $1 + \xi/\alpha$, and we see that for any fixed value of α there is a close to linear growth as a function of t_0. For large values of t_0 the dominating behavior is like $2 + 2\xi^2 t_0 + \xi^2/(\alpha(\alpha + \xi))$, which for $\alpha \to 0$ is like $2 + 2\xi^2 t_0 + \xi/\alpha$.

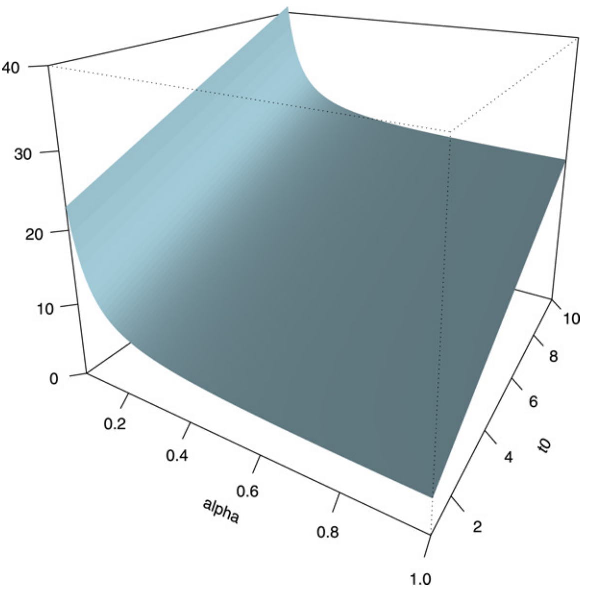

Fig. 1. With $g(t) = \min\{0, -\alpha(t - t_0)\}$ the figure shows the factor $\mathbb{E}^*(\exp(\theta^* L_\infty))$ as a function of α and t_0 for Brownian motion with drift -1 and variance 1.

4.4. Storage models

To illustrate some of the interpretations of the reflected Lévy process we consider in this section a setup with a container or storage that contains a (continuous) quantity, and we assume that initially the amount of the quantity is $x > 0$. We will assume that there is a flow out of the storage given by a subordinator $(B_t)_{t\geq 0}$, and that there is a flow into the storage given by another, independent subordinator $(A_t)_{t\geq 0}$. Defining the Lévy process $X_t = x + A_t - B_t$ then up to $\tau_0 = \inf\{t \geq 0 \mid X_t = 0\}$, X_t is a model of the quantity stored at time t. Reflection at 0 (i.e. modification of the flow out to be zero whenever $X_t = 0$) provides a process that is a valid storage model for all $t \geq 0$. We will not pursue this direction, but rather reflect the process at an upper barrier. Thus we introduce an upper bound on the capacity of the storage given in terms of a càdlàg function $h : [0, \infty) \to [0, \infty)$ with $h(0) = 0$ by limiting the maximal capacity of the storage to $x + h(t)$ at time $t \geq 0$. If something flows into the container when it is filled to its maximum capacity it immediately "overflows" and the overflow is lost. Consequently the total amount in the storage at time t (before the storage becomes empty for the first time) is

$$W_t^h = X_t - L_t$$

where $L_t = -\inf_{0\leq s\leq t}\{x + h(s) - X_s\} = \sup_{0\leq s\leq t}\{-h(s) - (B_s - A_s)\}$.

Clearly $W_t := x - W_t^h = B_t - A_t + L_t$ is the reflection of the Lévy process $(B_t - A_t)_{t\geq 0}$ at the barrier $g(t) := -h(t)$ for $t \geq 0$. Note that $(W_t)_{t\geq 0}$ does not depend upon the initial value x. We can also observe that

$$\tau = \inf\{t \geq 0 \mid W_t^h \leq 0\} = \inf\{t \geq 0 \mid W_t \geq x\}.$$

Thus $\mathbb{P}(\tau < \infty) = \mathbb{P}(\mathcal{M} > x)$ where $\mathcal{M} = \sup_{t\geq 0} W_t$.

The construction presented is a straightforward modification of the classical storage model with limited capacity but with the modification that the upper bound on the storage is allowed to be time dependent. With a queuing interpretation where the storage contains the residual workload in the queue this means that the queue buffer has an upper limit that changes with time and τ is the time to the first idle period. A water reservoir with a flow in and out and a time-changing capacity provides perhaps a better interpretation. Below we treat in detail several more concrete examples within the general framework presented above.

4.4.1. The Cramér–Lundberg risk process with a barrier

We can consider the classical Cramér–Lundberg risk process but reflected at the upper barrier $x + h(t)$. That is, we take $A_t = at$ with $a > 0$ and

$$B_t = \sum_{i=1}^{N_t} \xi_i,$$

where $(N_t)_{t\geq 0}$ is a Poisson process with intensity λ and $(\xi_n)_{n\geq 1}$ is a sequence of i.i.d. positive random variables with distribution function H independent of the Poisson process. We find that

$$\psi(\theta) = -a\theta + \log \mathbb{E}\left(\mathrm{e}^{\theta B_1}\right) = -a\theta - \lambda \int_0^\infty (1 - \mathrm{e}^{\theta x}) H(\mathrm{d}x),$$

is the Laplace exponent for $(B_t - at)_{t\geq 0}$, which we need to assume finite for some $\theta \geq 0$, and we assume that θ^* is a solution to $\psi(\theta) = 0$. We can interpret the process $x + at - B_t$ as the capital of an insurance company at time t (before ruin, i.e. for $t \leq \tau_0$) with initial capital x. We can then interpret W_t^h as the capital when we continuously pay out dividends. The function h represents a strategy for increasing (or decreasing) the capital reserve in the company by paying out dividend (the overflow) in the future. If $h(t) = \alpha t$, $\alpha > 0$, then since $(B_t - at)_{t\geq 0}$ is spectrally positive the results from Section 4.1 imply that

$$\mathbb{E}^*\left(\mathrm{e}^{\theta^* L_\infty}\right) = \frac{\theta_0}{\theta_0 - \theta^*}$$

where $\theta_0 > \theta^*$ solves

$$(a - \alpha)\theta = a\theta^* + \lambda \int_0^\infty (1 - \mathrm{e}^{(\theta^* - \theta)x}) H(\mathrm{d}x).$$

4.4.2. Modified M/M/1 queue with a linear barrier

We consider here $A_t = \sum_{i=1}^{N_t} \xi_i$ where $(N_t)_{t\geq 0}$ is a Poisson process with intensity λ, and we assume that ξ_n is exponentially distributed with parameter $\gamma > 0$. We also assume that $h(t) = \alpha t$ for $\alpha > 0$. If $B_t = t$ we have an M/M/1 queue with an upper bound on the buffer capacity that grows linearly in time. We will, however, allow for a general, stochastic "service rate" given by a subordinator $(B_t)_{t\geq 0}$ — though we need to assume that the Laplace exponent, $\psi_B(\theta)$, for $(B_t)_{t\geq 0}$ is finite for suitable $\theta > 0$. The Laplace exponent for $(B_t - A_t)_{t\geq 0}$ equals

$$\psi(\theta) = \psi_B(\theta) + \lambda \left(\frac{\gamma}{\gamma + \theta} - 1\right).$$

The Laplace exponent for $(-\alpha t + A_t - B_t)_{t\geq 0}$ under $\mathbb{P}^*$ is

$$\psi^*(\theta) = \psi(\theta^* - \theta) - \alpha\theta = \psi_B(\theta^* - \theta) + \lambda\left(\frac{\gamma}{\gamma + \theta^* - \theta} - 1\right) - \alpha\theta$$
$$= \psi_B(\theta^* - \theta) - \psi_B(\theta^*) + \frac{\lambda\gamma}{\gamma + \theta^*}\left(\frac{\gamma + \theta^*}{\gamma + \theta^* - \theta} - 1\right) - \alpha\theta$$

and the positive jumps remain exponential but with parameter $\gamma + \theta^*$ under $\mathbb{P}^*$. We observe that L_∞ equals the global maximum of a random walk with increments having distribution as $Z := \xi_1 - B_T - \alpha T$ where T is the time for the first jump of $(A_t)_{t\geq 0}$. According to Theorem VIII.5.8(b) in [2] the distribution of L_∞ is defective exponential, that is,

$$\mathbb{P}^*(L_\infty > x) = \rho\exp(-\eta x) \quad \text{and} \quad \mathbb{P}^*(L_\infty = 0) = 1 - \rho$$

where $\eta > 0$ solves $\mathbb{E}^*(\exp(\eta Z)) = 1$, or equivalently $\psi^*(\eta) = 0$, and where

$$\rho = 1 - \frac{\eta}{\gamma + \theta^*} = \frac{\gamma + \theta^* - \eta}{\gamma + \theta^*}.$$

In particular, introducing $\zeta = \eta - \theta^* > 0$, we find that

$$\mathbb{E}^*\left(e^{\theta^* L_\infty}\right) = (1 - \rho) + \rho\frac{\eta}{\eta - \theta^*} = 1 + \rho\left(\frac{\theta^*}{\eta - \theta^*}\right)$$
$$= 1 + \frac{\theta^*(\gamma - \zeta)}{(\gamma + \theta^*)\zeta} \tag{9}$$

where $\theta^* > 0$ solves $\psi(\theta) = 0$ and $\zeta > 0$ solves $\psi(-\zeta) - \alpha\zeta = \alpha\theta^*$. In greater generality, one can obtain similar results when the positive jumps are of phase-type relying on [16].

4.4.3. Modified M/M/1 queue with a general, non-decreasing barrier

We consider the same setup as above with $A_t = \sum_{i=1}^{N_t} \xi_i$, with $(N_t)_{t\geq 0}$ a Poisson process with intensity λ and ξ_n exponentially distributed with parameter $\gamma > 0$, but we allow for $h(t)$ to be a general, non-decreasing upper bound on the buffer capacity. Thus $g(t) = -h(t)$ is non-increasing. In this case we will first derive a formula for $\mathbb{E}^*(\exp(\theta^* L_\infty))$ using (8). The process $(L_t)_{t\geq 0}$ is then a pure jump process, and if we let $(J_n)_{n\geq 1}$ denote the jumps and $(R_t)_{t\geq 0}$ the corresponding counting process of jump times we find that

$$\tilde{L}_t = \sum_{i=1}^{R_t}\left[1 - e^{-\theta^* J_i}\right].$$

Since ξ_n is exponentially distributed with parameter $\gamma > 0$ we have, due to the memoryless property of the exponential distribution, that the jumps $(J_n)_{n\geq 0}$ will also form an i.i.d. sequence of exponentially distributed random variables with parameter γ, and of equal importance this sequence is independent of $(R_t)_{t\geq 0}$. Using (8) we arrive at the expression

$$\mathbb{E}^*\left(e^{\theta^* L_\infty}\right) = 1 + \mathbb{E}(R_\tau)\mathbb{E}\left(1 - e^{-\theta^* J_1}\right) = 1 + \mathbb{E}(R_\tau)\frac{\theta^*}{\gamma + \theta^*}$$

where τ is an independent random variable with distribution function $F_g(t) = 1 - \exp(\theta^* g(t))$.

We can also use this result to get the explicit formula for the linear barrier. Thus if we further specialize to a linear barrier, $h(t) = \alpha t$, we may observe that due to the strong Markov property of the Lévy process $(-\alpha t + A_t - B_t)_{t\geq}$, the interarrival times, $(T_n)_{n\geq 0}$, for the new maxima of the process form an i.i.d. sequence making $(R_t)_{t\geq 0}$ a renewal process (it may be terminating, in which case we ignore the interarrival times after the first that equals ∞). We have $(R_t > n) = (\sum_{i\leq n+1} T_i \leq t)$ and with G the distribution function for T_1 we find that

$$\begin{aligned}\mathbb{E}(R_\sigma) &= \sum_{n=0}^{\infty}\int_0^\infty \mathbb{P}(R_t > n)F_g(\mathrm{d}t) = \sum_{n=0}^{\infty}\int_0^\infty G^{*(n+1)}(t)F_g(\mathrm{d}t)\\ &= \sum_{n=1}^{\infty}\int_0^\infty \exp(-\theta^*\alpha t)G^{*n}(\mathrm{d}t) = \sum_{n=1}^{\infty}\phi(-\theta^*\alpha)^n = \frac{1}{\phi(-\theta^*\alpha)^{-1}-1}\end{aligned}$$

where ϕ denotes the Laplace transform of G and where we used integration by parts for the third equality. When ξ_n is exponential with parameter γ and $g(t) = -\alpha t$ we thus obtain the representation

$$\mathbb{E}^*\left(\mathrm{e}^{\theta^* L_\infty}\right) = 1 + \frac{\theta^*}{(\phi(-\theta^*\alpha)^{-1}-1)(\gamma+\theta^*)}. \tag{10}$$

Direct calculation of $\phi(-\theta^*\alpha)$ is possible due to the positive jumps being exponential. Martin Jacobsen showed (personal communication) how to compute the relevant Laplace transform from Theorem 1 in [17] when the subordinator $(B_t)_{t\geq 0}$ has finitely many jumps in finite time, but the setup in [17] is in a number of ways considerably more general than needed. Noting that $\phi(-\alpha\theta^*) = \mathbb{P}(L_\sigma > 0)$ where σ is an independent exponentially distributed variable with parameter $\alpha\theta^*$ it follows directly from Theorem 2(B) in [16] that

$$\phi(-\theta^*\alpha) = 1 - \frac{\zeta}{\gamma}$$

where ζ solves the equation

$$\psi_B(-\zeta) + \lambda\left(\frac{\gamma}{\gamma-\zeta} - 1\right) - \alpha\zeta = \psi(-\zeta) - \alpha\zeta = \alpha\theta^*.$$

The result of combining this with (10) is, of course, in concordance with (9).

5. Proofs

Proof (Theorem 1). Fix $u > 0$ and define the stopping time

$$\tau^g(u) = \inf\{t \geq 0 \mid W_t > u\} \tag{11}$$

then since $W_t \geq X_t$ for all $t \geq 0$ and $X_t \to \infty$ $\mathbb{P}^*$-a.s. we have that $\mathbb{P}^*(\tau^g(u) < \infty) = 1$. From Theorem XIII.3.2 in [2] we have that

$$\begin{aligned}\mathbb{P}(\mathcal{M} > u) &= \mathbb{P}(\tau^g(u) < \infty) = \mathbb{E}^*(\mathrm{e}^{-\theta^* X_{\tau^g(u)}})\\ &= \mathrm{e}^{-\theta^* u}\mathbb{E}^*(\mathrm{e}^{-\theta^*(X_{\tau^g(u)}-u)}).\end{aligned} \tag{12}$$

Using the definition of W_t we see that

$$X_{\tau^g(u)} - u = W_{\tau^g(u)} - u - (W_{\tau^g(u)} - X_{\tau^g(u)}) = B_u - L_{\tau^g(u)}$$

where $B_u = W_{\tau^g(u)} - u \geq 0$. Since $(L_t)_{t \geq 0}$ is non-decreasing we have that $L_{\tau^g(u)} \leq L_\infty$ and since $\exp(-\theta^* B_u) \leq 1$ we conclude that

$$\mathbb{P}(\mathcal{M} > u) \leq \mathrm{e}^{-\theta^* u} \mathbb{E}^*(\mathrm{e}^{\theta^* L_\infty}). \tag{13}$$

If $\mathbb{E}^*(\exp(\theta^* L_\infty)) < \infty$ then (13) shows that $\mathbb{P}(\mathcal{M} < \infty) = 1$. Suppose, on the contrary, that $\mathbb{P}(\mathcal{M} = \infty) > 0$, then (13) implies that for all $u \geq 0$

$$\mathrm{e}^{\theta^* u} \mathbb{P}(\mathcal{M} = \infty) \leq \mathbb{E}^*(\mathrm{e}^{\theta^* L_\infty})$$

where the left hand side converges to ∞ for $u \to \infty$, hence $\mathbb{E}^*(\exp(\theta^* L_\infty)) = \infty$. □

Remark 8. Note that if $(X_t)_{t \geq 0}$ is a spectrally negative Lévy processes we have for $u > 0$ that $W_{\tau^g(u)} - u = 0$. This follows from the simple observation that jumps upwards above a positive level for the reflected process can only be due to upward jumps in the Lévy process. This observation shows that $X_{\tau^g(u)} - u = -L_{\tau^g(u)}$ and since $\tau^g(u) \to \infty$ $\mathbb{P}^*$-a.s. for $u \to \infty$ we get from (12) by monotone convergence that

$$\lim_{u \to \infty} \mathrm{e}^{\theta^* u} \mathbb{P}(\mathcal{M} > u) = \mathbb{E}^*(\mathrm{e}^{\theta^* L_\infty}).$$

This proves Theorem 2 for the special case of spectrally negative Lévy processes.

The only obstacle in generalizing the proof of Theorem 2 given in the remark above to Lévy processes that are also allowed to have positive jumps is to be able to handle the overshoot above a high level properly. To this end we need the following technical lemma.

Lemma 9. *With $\tau^g(u)$ defined by (11) and with*

$$h : [0, \infty) \to \mathbb{R}$$

a bounded, continuous function, then given the non-lattice assumption of Theorem 2 and with B_∞ the random variable given by (1) it holds that

$$\mathbb{E}^* \left| \mathbb{E}^* \left(h(W_{\tau^g(u)} - u) \mid \mathcal{F}_{\tau^g(u/2)} \right) - \mathbb{E}^*(h(B_\infty)) \right| \to 0 \tag{14}$$

for $u \to \infty$.

Proof. For convenience extend h to be defined on $\mathbb{R}$ by $h(u) = 0$ for $u < 0$. Let $X = (X_t)_{t \geq 0}$, $X' = (X_{\tau^g(u/2)+t} - X_{\tau^g(u/2)})_{t \geq 0}$ and $Y = u - W_{\tau^g(u/2)}$. By the strong Markov property for Lévy processes we see that X and X' have the same distribution and that X' is independent of $\mathcal{F}_{\tau^g(u/2)}$ (under $\mathbb{P}^*$). Moreover, Y is clearly $\mathcal{F}_{\tau^g(u/2)}$ measurable. Recall the definition $\tau(u) = \inf\{n \geq 0 \mid X_t > u\}$ and define

$$\sigma(u) = \inf\{t \geq 0 \mid X_{\tau^g(u/2)+t} - X_{\tau^g(u/2)} > u - W_{\tau^g(u/2)}\}.$$

Then with

$$H(u) = \mathbb{E}^*(h(X_{\tau(u)} - u))$$

it follows that $\mathbb{E}^* \left(h(X'_{\sigma(u)} - Y) | \mathcal{F}_{\tau^g(u/2)} \right) = H(Y)$, or written out

$$\mathbb{E}^* \left(h \left(X_{\tau^g(u/2)+\sigma(u)} - X_{\tau^g(u/2)} - \left(u - W_{\tau^g(u/2)} \right) \right) \mid \mathcal{F}_{\tau^g(u/2)} \right) = H \left(u - W_{\tau^g(u/2)} \right)$$

From (1) it follows that $H(u) \to \mathbb{E}^*(h(B_\infty))$ for $u \to \infty$. Since

$$0 \leq W_{\tau^g(u)} - u = X_{\tau^g(u)} - u + L_{\tau^g(u)} \leq X_{\tau(u)} - u + L_\infty,$$

where the r.h.s. is $\mathbb{P}^*$-tight due to (1), we find that

$$u - W_{\tau^g(u/2)} = u/2 - (W_{\tau^g(u/2)} - u/2) \xrightarrow{\mathbb{P}^*} \infty.$$

We conclude that

$$\mathbb{E}^* \left| H(u - W_{\tau^g(u/2)}) - \mathbb{E}^*(h(B_\infty)) \right| \to 0 \tag{15}$$

for $u \to \infty$.

Recall that

$$L_{\tau^g(u)} = W_{\tau^g(u)} - X_{\tau^g(u)} = \sup_{0 \le s \le \tau^g(u)} \{\max\{g(s) - X_s, 0\}\}$$

and note that since $\tau^g(u) \to \infty$ $\mathbb{P}^*$-a.s. for $u \to \infty$ it follows that $L_{\tau^g(u)} = L_\infty$ eventually with $\mathbb{P}^*$-probability one. Letting $K_u = (L_{\tau^g(u/2)} = L_\infty)$ then $1(K_u^c) \to 0$ for $u \to \infty$ $\mathbb{P}^*$-a.s. and in particular $\mathbb{P}^*(K_u^c) \to 0$ for $u \to \infty$. On the event K_u it holds that $\tau^g(u) = \tau^g(u/2) + \sigma(u)$ and that $W_{\tau^g(u/2)} - X_{\tau^g(u/2)} = L_{\tau^g(u/2)} = L_{\tau^g(u)} = W_{\tau^g(u)} - X_{\tau^g(u)}$. In particular, on K_u

$$X_{\tau^g(u/2)+\sigma(u)} - X_{\tau^g(u/2)} = X_{\tau^g(u)} - X_{\tau^g(u/2)} = W_{\tau^g(u)} - W_{\tau^g(u/2)}.$$

Then

$$\begin{aligned} &\mathbb{E}^* \left| \mathbb{E}^* \left(h(W_{\tau^g(u)} - u) \mid \mathcal{F}_{\tau^g(u/2)} \right) - H(u - W_{\tau^g(u/2)}) \right| \\ &\quad \le \mathbb{E}^* \left(\left| h(W_{\tau^g(u)} - u) - h\left(X_{\tau^g(u/2)+\sigma(u)} - X_{\tau^g(u/2)} - \left(u - W_{\tau^g(u/2)} \right) \right) \right| 1(K_u^c) \right) \\ &\quad \le 2\|h\|_\infty \mathbb{P}^*(K_u^c) \to 0 \end{aligned}$$

and this together with (15) completes the proof. $\square$

Proof (Theorem 2). Using the notation $B_u = W_{\tau^g(u)} - u$ as in the proof of Theorem 1 we obtain from (12) the equality

$$\mathbb{P}(\mathcal{M} > u) = \mathrm{e}^{-\theta^* u} \mathbb{E}^* \left(\mathrm{e}^{-\theta^* B_u} \mathrm{e}^{\theta^* L_{\tau^g(u)}} \right). \tag{16}$$

With $K_u = (L_{\tau^g(u/2)} = L_\infty)$ as in the proof of Lemma 9 we have that $L_{\tau^g(u/2)} = L_{\tau^g(u)}$ on K_u, and since $B_u \ge 0$ and $L_{\tau^g(u/2)} \le L_{\tau^g(u)} \le L_\infty$ we see that

$$\begin{aligned} \mathbb{E}^* \left| \mathrm{e}^{-\theta^* B_u} \mathrm{e}^{\theta^* L_{\tau^g(u)}} - \mathrm{e}^{-\theta^* B_u} \mathrm{e}^{\theta^* L_{\tau^g(u/2)}} \right| &= \mathbb{E}^* \left(\mathrm{e}^{-\theta^* B_u} \left| \mathrm{e}^{\theta^* L_{\tau^g(u)}} - \mathrm{e}^{\theta^* L_{\tau^g(u/2)}} \right| 1(K_u^c) \right) \\ &\le \mathbb{E}^* \left(\left(\mathrm{e}^{\theta^* L_{\tau^g(u)}} - \mathrm{e}^{\theta^* L_{\tau^g(u/2)}} \right) 1(K_u^c) \right) \\ &\le \mathbb{E}^* \left(\mathrm{e}^{\theta^* L_\infty} 1(K_u^c) \right) \to 0 \end{aligned}$$

for $u \to \infty$ where we use dominated convergence and the fact that $1(K_u^c) \to 0$ $\mathbb{P}^*$-a.s. as noted in the proof of Lemma 9. With $C_u = \exp(\theta^* L_{\tau^g(u/2)})$, $D_u = \mathbb{E}^*(\exp(-\theta^* B_u) \mid \mathcal{F}_{\tau^g(u/2)})$ and likewise $C_\infty = \exp(\theta^* L_\infty)$ and $D_\infty = \mathbb{E}^*(\exp(-\theta^* B_\infty))$ we have

$$C_u D_u - C_\infty D_\infty = C_u(D_u - D_\infty) + D_\infty(C_u - C_\infty).$$

Now, since $|D_\infty| \le 1$, $\mathbb{E}^*|D_\infty(C_u - C_\infty)| \le \mathbb{E}^*(C_\infty) - \mathbb{E}^*(C_u) \to 0$ by monotone convergence since $C_u \nearrow C_\infty$ $\mathbb{P}^*$-a.s. and $\mathbb{E}^*(C_\infty) < \infty$ by assumption. Moreover, $D_u, D_\infty \in [0, 1]$, so $|D_u - D_\infty| \le 1$ and since $C_u \le C_\infty$ we find for any $K > 0$

$$\begin{aligned} \mathbb{E}^*|C_u(D_u - D_\infty)| &\le \mathbb{E}^*|C_\infty(D_u - D_\infty)| \\ &\le K\mathbb{E}^*|D_u - D_\infty| + \mathbb{E}^*(C_\infty 1(C_\infty \ge K)). \end{aligned}$$

By Lemma 9 with $h(x) = \exp(-\theta^* x)$ we have $\mathbb{E}^*|D_u - D_\infty| \to 0$ for $u \to \infty$, hence

$$\limsup_{u\to\infty} \mathbb{E}^*|C_u(D_u - D_\infty)| \leq \mathbb{E}^*(C_\infty 1(C_\infty \geq K)) \to 0$$

for $K \to \infty$, again since $\mathbb{E}^*(C_\infty) < \infty$ by assumption. In conclusion we have $\mathbb{E}^*|C_u D_u - C_\infty D_\infty| \to 0$ for $u \to \infty$, and in particular

$$\begin{aligned}\lim_{u\to\infty} \mathbb{E}^* \left(e^{\theta^* L_{\tau^g(u/2)}} \mathbb{E}^* \left(e^{-\theta^* B_u} | \mathcal{F}_{\tau^g(u/2)} \right)\right) &= \mathbb{E}^* \left(e^{\theta^* L_\infty} \mathbb{E}^*(e^{-\theta^* B_\infty}) \right) \\ &= \mathbb{E}^* \left(e^{-\theta^* B_\infty} \right) \mathbb{E}^* \left(e^{\theta^* L_\infty} \right).\end{aligned}$$

Collecting the observations we find that

$$\begin{aligned}\mathbb{E}^* \left(e^{-\theta^* B_\infty} \right) \mathbb{E}^* \left(e^{\theta^* L_\infty} \right) &= \lim_{u\to\infty} \mathbb{E}^* \left(e^{\theta^* L_{\tau^g(u/2)}} \mathbb{E}^* \left(e^{-\theta^* B_u} | \mathcal{F}_{\tau^g(u/2)} \right)\right) \\ &= \lim_{u\to\infty} \mathbb{E}^* \left(e^{-\theta^* B_u} e^{\theta^* L_{\tau^g(u/2)}} \right) \\ &= \lim_{u\to\infty} \mathbb{E}^* \left(e^{-\theta^* B_u} e^{\theta^* L_{\tau^g(u)}} \right),\end{aligned}$$

and this completes the proof. □

Proof (*Version 1 of* Theorem 3). This proof – based on stochastic analysis – revolves around a martingale decomposition of the process $(\exp(\theta^* W_t))_{t\geq 0}$. An application of integration by parts, Proposition I.4.49(a) in [12], gives

$$e^{\theta^* W_t} = e^{\theta^* X_t} e^{\theta^* L_t} = 1 + e^{\theta^* X_t} \bullet e^{\theta^* L_t} + e^{\theta^* L_{t-}} \bullet e^{\theta^* X_t},$$

where the last term, $M_t := \exp(\theta^* L_{t-}) \bullet \exp(\theta^* X_t)$, forms a local martingale since the integrator is a martingale and the integrand is a locally bounded, predictable process, see I.4.34(b) in [12]. The second term is a pathwise Lebesgue–Stieltjes integral, and $\exp(\theta^* L_t) = 1 + \exp(\theta^* L_t) \bullet \tilde{L}_t$, cf. Remark 4. Then noting that the support of the measure given by $(\tilde{L}_t)_{t\geq 0}$ coincides with the support of $(L_t)_{t\geq 0}$, which is contained in the set $\{t \geq 0 \mid W_t = g(t)\}$, we find that

$$e^{\theta^* X_t} \bullet e^{\theta^* L_t} = e^{\theta^*(X_t + L_t)} \bullet \tilde{L}_t = e^{\theta^* W_t} \bullet \tilde{L}_t = e^{\theta^* g(t)} \bullet \tilde{L}_t.$$

Therefore

$$e^{\theta^* W_t} = 1 + e^{\theta^* g(t)} \bullet \tilde{L}_t + M_t = 1 + \int_0^t e^{\theta^* g(s)} \, d\tilde{L}_s + M_t.$$

The local martingale $(-M_t)_{t\geq 0}$ is (a special version of) the *Kella–Whitt* martingale, see Theorem 2 in [11] or Theorem IX.3.1 in [2]. Theorem 3 follows if one can show that the local martingale is a true martingale, but in this case there is a workaround so that the local martingale does not need to be a martingale at all. Taking $(T_n)_{n\geq 0}$ to be any localizing sequence for $(M_t)_{t\geq 0}$, that is, an increasing sequence of stopping times that tend to ∞ a.s., we can due to positivity always conclude that

$$\mathbb{E}\left(e^{\theta^* W_{t\wedge T_n}} \right) = 1 + \mathbb{E}\left(\int_0^{t\wedge T_n} e^{\theta^* g(s)} \, d\tilde{L}_s \right).$$

For the left hand side we use the Esscher transform, cf. Theorem XIII.3.2 in [2], and the fact that $(L_t)_{t\geq 0}$ is increasing together with monotone convergence to conclude that

$$\mathbb{E}\left(e^{\theta^* W_{t\wedge T_n}} \right) = \mathbb{E}^* \left(e^{\theta^* L_{t\wedge T_n}} \right) \to \mathbb{E}^* \left(e^{\theta^* L_t} \right)$$

for $n \to \infty$. For the right hand side we can use monotone convergence directly as $n \to \infty$, and we conclude that

$$\mathbb{E}^*\left(\mathrm{e}^{\theta^* L_t}\right) = 1 + \mathbb{E}\left(\int_0^t \mathrm{e}^{\theta^* g(s)}\,\mathrm{d}\tilde{L}_s\right).$$

A second application of monotone convergence on both sides as $t \to \infty$ yields

$$\mathbb{E}^*\left(\mathrm{e}^{\theta^* L_\infty}\right) = 1 + \mathbb{E}\left(\int_0^\infty \mathrm{e}^{\theta^* g(s)}\,\mathrm{d}\tilde{L}_s\right). \quad \square$$

The proof above, called version 1, of Theorem 3 provides insight into how the theorem relates to well established results and methods such as the Kella–Whitt martingale and stochastic exponentials and logarithms. It relies on the other hand quite extensively on the machinery of stochastic analysis. The following proof is elementary relying on nothing but classical integration theory.

Proof (*Version 2 of* Theorem 3). By partial integration we have that

$$\mathbb{E}^*\left(\mathrm{e}^{\theta^* L_\infty}\right) = \int_{-\infty}^\infty \theta^* \mathrm{e}^{\theta^* u}\mathbb{P}^*(L_\infty > u)\,\mathrm{d}u.$$

If we introduce $\sigma(u) = \inf\{t > 0 \mid L_t > u\}$ we find that $\mathbb{P}^*(L_\infty > u) = \mathbb{P}^*(\sigma(u) < \infty) = \mathbb{E}(\exp(\theta^* X_{\sigma(u)}); \sigma(u) < \infty)$. Next observe that $\sigma(u) = 0$ for $u \leq 0$ (and thus $X_{\sigma(u)} = 0$), and $L_{\sigma(u)} = g(\sigma(u)) - X_{\sigma(u)}$ for $u > 0$ by the definition of $(L_t)_{t\geq 0}$. These identities, and an application of Tonelli, provide the formula

$$\begin{aligned}\mathbb{E}^*\left(\mathrm{e}^{\theta^* L_\infty}\right) &= \int_{-\infty}^0 \theta^* \mathrm{e}^{\theta^* u}\,\mathrm{d}u + \mathbb{E}\left(\theta^* \int_0^\infty 1(\sigma(u) < \infty)\,\mathrm{e}^{\theta^*[u + g(\sigma(u)) - L_{\sigma(u)}]}\,\mathrm{d}u\right)\\ &= 1 + \mathbb{E}\left(\theta^* \int_0^\infty 1(\sigma(u) < \infty)\mathrm{e}^{\theta^*[u + g(\sigma(u)) - L_{\sigma(u)}]}\,\mathrm{d}u\right).\end{aligned}$$

To proceed we rewrite the inner integral on the r.h.s. above using a pathwise decomposition of the integration interval $[0, \infty)$ according to whether $\Delta L_{\sigma(u)} > 0$ or $= 0$. Thus

$$\begin{aligned}&\theta^* \int_0^\infty 1(\sigma(u) < \infty)\mathrm{e}^{\theta^*[u + g(\sigma(u)) - L_{\sigma(u)}]}\,\mathrm{d}u\\ &= \theta^* \int 1_{\{t \mid \Delta L_t > 0\}} \circ \sigma(u)\mathrm{e}^{\theta^*[u + g(\sigma(u)) - L_{\sigma(u)}]}\,\mathrm{d}u \qquad (17)\\ &\quad + \theta^* \int 1_{\{t \mid \Delta L_t = 0\}} \circ \sigma(u)\mathrm{e}^{\theta^* g(\sigma(u))}\,\mathrm{d}u \qquad (18)\end{aligned}$$

where we have used that when $\Delta L_{\sigma(u)} = 0$ we have $L_{\sigma(u)} = u$. Noting that Lebesgue almost surely

$$1_{\{t \mid \Delta L_t > 0\}} \circ \sigma(u) \overset{\text{a.s.}}{=} \sum_{t:\Delta L_t > 0} 1_{[L_{t-}, L_t]}(u),$$

integral (17) can be rewritten as

$$\theta^* \int 1_{\{t \mid \Delta L_t > 0\}} \circ \sigma(u)\mathrm{e}^{\theta^*[u + g(\sigma(u)) - L_{\sigma(u)}]}\,\mathrm{d}u = \sum_{t:\Delta L_t > 0} \theta^* \int_{L_{t-}}^{L_t} \mathrm{e}^{\theta^*[u + g(t) - L_t]}\,\mathrm{d}u$$

$$= \sum_{t:\Delta L_t > 0} e^{\theta^* g(t)} e^{-\theta^* L_t} (e^{\theta^* L_t} - e^{\theta^* L_{t-}})$$
$$= \sum_{t \geq 0} e^{\theta^* g(t)} (1 - e^{-\theta^* \Delta L_t}).$$

To rewrite (18) we note that with λ the Lebesgue measure on $[0, \infty]$ the transformed measure $\sigma(\lambda)$ is the Lebesgue–Stieltjes measure given by the increasing, right-continuous function $(L_t)_{t \geq 0}$. An application of the integral transformation theorem therefore gives

$$\int 1_{\{t|\Delta L_t = 0\}} \circ \sigma(u)\, e^{\theta^* g(\sigma(u))}\, du = \int 1_{\{t|\Delta L_t = 0\}}(s)\, e^{\theta^* g(s)}\, dL_s = \int e^{\theta^* g(s)}\, dL_s^c$$

and this completes the proof. □

The following Lemma provides a technical bound on $\mathbb{E}(\tilde{L}_t)$, which we need to give a proof of Corollary 5. As the proof will show, the bound is rather trivial under the assumption $\mathbb{E}|X_1| < \infty$ but a little more sophistication is needed if we relax the finite first moment assumption and instead require g to be non-increasing.

Lemma 10. *If either $\mathbb{E}|X_1| < \infty$ or g is non-increasing we have*

$$\mathbb{E}(\tilde{L}_t) \leq ct + b$$

for some constants $b, c \in (0, \infty)$.

Proof. If $\mathbb{E}|X_1| < \infty$ then with $\xi = \mathbb{E}(X_1)$ it follows directly by the definition of $\tilde{L}_t$ and L_t that

$$\frac{1}{\theta^*} \tilde{L}_t \leq L_t \leq \sup_{0 \leq s \leq t} |X_s| \leq |\xi| t + \sup_{0 \leq s \leq t} |X_s - \xi s|.$$

From Theorem VII.5.1 in [18] – see also [19], Theorem 25.18 and Remark 25.19 – we find that

$$\mathbb{E}\left(\sup_{0 \leq s \leq t} |X_s - \xi s| \right) \leq 8\mathbb{E}|X_t - \xi t| \leq 8(t+1)\mathbb{E}|X_1 - \xi|,$$

which gives the desired result.

If g is assumed non-increasing, we can apply a different argument, that does not rely on finiteness of the first moment of X_1. By the definition of $\tilde{L}_t$ and monotonicity of g we get for any $t, s \geq 0$

$$\frac{1}{\theta^*}(\tilde{L}_{t+s} - \tilde{L}_t) \leq L_{t+s} - L_t \leq \sup_{0 \leq s' \leq s} \{g(t+s') - g(t) - X_{t+s'} + X_t\} \leq \sup_{0 \leq s' \leq s} \{X_t - X_{t+s'}\}. \tag{19}$$

Note that the monotonicity assumption on g is crucial for this inequality, and if we allow for non-monotonicity of g the whole argument below breaks down. A Lévy–Itô type of decomposition corresponding to the restriction of the Lévy measure to $[-1, \infty)$ and $(-\infty, -1)$ respectively allows us to write

$$X_t = X_t^{(1)} + X_t^{(2)}$$

where $(X_t^{(1)})_{t\geq 0}$ is a Lévy process whose negative jumps are of absolute size ≤ 1 and where $(X_t^{(2)})_{t\geq 0}$ is an independent compound Poisson process with negative jumps of absolute size ≥ 1. We let $(T_i)_{i\geq 1}$ denote the jump times for the compound Poisson process $(X_t^{(2)})_{t\geq 0}$ (and $T_0 = 0$) and we let $N_t = \sum_{i\geq 0} 1(T_i \leq t)$. With $Y_i = \sup_{0\leq s\leq T_{i+1}-T_i}\{X_{T_i}^{(1)} - X_{T_i+s}^{(1)}\}$ for $i \geq 0$ we claim that

$$\tilde{L}_t \leq L'_t := \sum_{i=0}^{N_t}(1 + \theta^* Y_i).$$

Indeed, for $t = T_0 = 0$ the inequality holds. If the inequality holds for $t \in [0, T_n)$, we note that $\tilde{L}_{T_n} \leq L'_{T_n-} + 1$ because $\tilde{L}_t$ has jumps of at most size 1. Then for $t \in [T_n, T_{n+1})$, i.e. $N_t = n+1$, we have that

$$\begin{aligned}\tilde{L}_t &= \tilde{L}_{T_n} + \tilde{L}_{T_n+(t-T_n)} - \tilde{L}_{T_n} \leq L'_{T_n-} + 1 + \theta^* \sup_{0\leq s'\leq (t-T_n)}\{X_{s'} - X_{T_n+s'}\}\\ &\leq \sum_{i=1}^{N_t-1}(1 + \theta^* Y_i) + 1 + \theta^* Y_{n+1} = L'_t.\end{aligned}$$

Here the second inequality follows from (19) and the last inequality follows from the fact that on the interval $[T_n, T_{n+1})$ the Lévy process has no negative jumps of absolute size ≥ 1. Since $N_t - 1$ is a homogeneous Poisson process independent of $(Y_i)_{i\geq 0}$ and since the latter is a sequence of i.i.d. positive, random variables we get that

$$\mathbb{E}(\tilde{L}_t) \leq \mathbb{E}(N_t)(1 + \theta^*\mathbb{E}(Y_0)) = (\mathbb{E}(N_1 - 1)t + 1)(1 + \theta^*\mathbb{E}(Y_0)).$$

Given that $\mathbb{E}(Y_0) < \infty$, this gives the desired result. Under the Cramér condition, the Lévy measure restricted to $[1, \infty)$ has moments of all orders and the Lévy process $(X_t^{(1)})_{t\geq 0}$ has in particular finite first moment, cf. Exercise 2.6 in [1]. Thus as $\mathbb{E}|X_1^{(1)}| < \infty$ we can refer to Doob as above but with the running maximum not up to a fixed time but up to an independent exponentially distributed time, and this gives that also Y_0 has finite first moment. □

Proof (Corollary 5). If $F_g(t) = 1 - \exp(\theta^* g(t))$ has locally bounded variation, integration by parts yields that

$$\mathbb{E}\left(\int_0^t e^{\theta^* g(s)}\,d\tilde{L}_s\right) = e^{\theta^* g(t)}\mathbb{E}(\tilde{L}_t) + \int_0^t \mathbb{E}(\tilde{L}_{s-}).F_g(ds). \tag{20}$$

Due to the assumption that $\int_0^\infty tF_g^+(dt) < \infty$ we conclude, using Lemma 10, that

$$e^{\theta^* g(t)}\mathbb{E}(\tilde{L}_t) \leq (ct + b)F_g(t, \infty) \leq (ct + b)F_g^+(t, \infty) \to 0$$

for $t \to \infty$. Since $0 \leq \exp(\theta^* g(t)) = F_g^+(t, \infty) - F_g^-(t, \infty)$ the assumption $\int_0^\infty tF_g^+(dt) < \infty$ implies that also $\int_0^\infty tF_g^-(dt) < \infty$. Due to Lemma 10 we conclude that $\mathbb{E}(\tilde{L}_{t-})$ is F_g-integrable. Using monotone convergence for the positive and negative parts separately we obtain

$$\int_0^t \mathbb{E}(\tilde{L}_{s-})F_g(ds) \to \int_0^\infty \mathbb{E}(\tilde{L}_{s-})F_g(ds) < \infty$$

for $t \to \infty$. The left hand side in (20) converges monotonely and we conclude that

$$\mathbb{E}\left(\int_0^\infty e^{\theta^* g(s)}\,d\tilde{L}_s\right) = \int_0^\infty \mathbb{E}(\tilde{L}_{s-})F_g(ds). \quad \square$$

6. Concluding remarks

The subject matter of this paper is the reflection of a Lévy process at a *deterministic* barrier. It would obviously also be of interest to allow the barrier to be stochastic. For the easy case where the barrier is independent of the Lévy process, we can simply condition on the barrier. A result like Theorem 2 then holds conditionally on the realization of the barrier, and only the factor $\mathbb{E}^*(\exp(\theta^* L_\infty))$ on the right hand side of the asymptotic expression depends upon the concrete realization. One can use Theorem 3 to check if that factor is finite almost surely, and thus whether Theorem 2 holds for almost all realizations of the barrier. If we allow for dependence of the barrier on the Lévy process the current paper does not seem to offer much insight in general.

Another possible direction for generalizations is to loosen the assumption of a light, positive tail. Then it seems natural to attempt combining [20] with the setup of the present paper.

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