A Useful Strengthening of the Stone-Weierstrass Theorem

Søren Boel, Toke Meier Carlsen, and Niels Richard Hansen

A well known version of the Stone-Weierstrass Theorem ([**3**, p. 146], [**5**, p. 122], [**6**] and [**8**, p. 8]) states that given a locally compact Hausdorff space X, and given a self-adjoint, point-separating subalgebra $\mathcal{A}$ of $C_0(X)$ that does not vanish identically at any point, then $\mathcal{A}$ is uniformly dense in $C_0(X)$.

As we show in the following theorem, a simple application of the Stone-Weierstrass Theorem allows us to conclude in such cases that not only do we have uniform approximation to any function in $C_0(X)$, we can also at the same time demand exact agreement at any finite number of points. This hybrid of pointwise and uniform approximation is sometimes a useful piece of information, even in the classical setting of Weierstrass with polynomials on an interval.

Theorem. Let X be a locally compact Hausdorff space and let $\mathcal{A}$ be a self-adjoint subalgebra of $C_0(X)$ that separates points in X and does not vanish identically at any point of X. Suppose that $f \in C_0(X)$, $x_1, \dots, x_n$ are points in X, and suppose $\epsilon > 0$. Then there exists a function p in $\mathcal{A}$ such that $p(x_i) = f(x_i),\ i = 1, \dots, n$ and $|p(x) - f(x)| < \epsilon,\ x \in X$.

This theorem follows directly from the Stone-Weierstrass Theorem and the Singer-Yamabe Theorem [**2**, p. 49], but we give a simple proof that does not involve the Singer-Yamabe Theorem or any other topological vector space theory. For the proof of the theorem, we need two lemmas:

Lemma 1. *If $x'_1, \dots, x'_m$ are distinct points in X, and if $y_1, \dots, y_m$ are (not necessarily distinct) complex numbers, then there exists a function p in $\mathcal{A}$ such that $p(x'_i) = y_i$, $i = 1, \dots, m$.*

Proof. Let $\Phi : C_0(X) \to \mathbb{C}^m$ be the evaluation homomorphism:

$$\Phi(f) = (f(x'_1), \dots, f(x'_m)),\ \ f \in C_0(X).$$

It follows directly from the Tietze Extension Theorem ([**1**, p. 99] or [**4**, p. 389]) that Φ is surjective. By the Stone-Weierstrass Theorem $\mathcal{A}$ is dense in $C_0(X)$, hence $\Phi(\mathcal{A})$ is dense in $\mathbb{C}^m$. But the only dense subspace of $\mathbb{C}^m$ is $\mathbb{C}^m$, so the conclusion follows immediately. ■

Lemma 2. *Consider the locally compact Hausdorff space*

$$\tilde{X} = X \setminus \{x_1, \dots, x_n\}$$

and the function algebra

$$\tilde{\mathcal{A}} = \{f|_{\tilde{X}} \mid f \in \mathcal{A},\ f(x_1) = \cdots = f(x_n) = 0\}.$$

Then $\tilde{\mathcal{A}}$ *is dense in* $C_0(\tilde{X})$.

Proof. Clearly $\tilde{\mathcal{A}}$ is a self-adjoint subalgebra of $C_0(\tilde{X})$. By Lemma 1 (applied with $m = n + 2$, respectively, $m = n + 1$), $\tilde{\mathcal{A}}$ separates points and does not vanish identically at any point of $\tilde{X}$. Hence $\tilde{\mathcal{A}}$ is dense in $C_0(\tilde{X})$ by the Stone-Weierstrass Theorem. ■

Proof of the Theorem. First use Lemma 1 to pick a function p_1 in $\mathcal{A}$, such that $p_1(x_i) = f(x_i)$, $i = 1, \ldots, n$. Then use Lemma 2 to pick a function p_2 in $\mathcal{A}$ such that $p_2(x_1) = \cdots = p_2(x_n) = 0$ and $|p_2(x) - (f(x) - p_1(x))| < \epsilon,\ x \in \tilde{X}$. Now $p = p_1 + p_2$ has the desired property. ■

REFERENCES

1. E. Hewitt and K. Stromberg, *Real and Abstract Analysis*, Grad. Texts in Math. **25**, Springer-Verlag, Berlin, 1965.
2. R. B. Holmes, *Geometric Functional Analysis and its Applications*, Grad. Texts in Math. **24**, Springer-Verlag, New York, 1975.
3. G. K. Pedersen, *Analysis Now*, Grad. Texts in Math. **118**, 2nd print., Springer-Verlag, Berlin, 1995.
4. W. Rudin, *Real and Complex Analysis*, 3rd ed., McGraw-Hill, Singapore, 1987.
5. W. Rudin, *Functional Analysis*, 2nd ed., McGraw-Hill, New York, 1991.
6. M. H. Stone, The generalized Weierstrass approximation Theorem, *Math. Mag.* **21** (1947–48) 167–184, 237–254.
7. K. Weierstrass, Über die analytische Darstellbarkeit sogenannter willkürlicher Funktionen einer reellen Veränderlichen, *Sitzungsberichte der Köninglich Preußischen Akademie der Wissenschaften zu Berlin* (1885) 633–639, 789–805.
8. K. Yosida, *Functional Analysis*, A Series of Comprehensive Stud. in Math. **123**, 6th ed., Springer-Verlag, Berlin, 1980.

University of Copenhagen, Universitetsparken 5, DK-2100, Copenhagen Ø, Denmark.
boel@math.ku.dk
toke@math.ku.dk
richard@math.ku.dk.

Linear Preservers that Permute the Entries of a Matrix

Xingzhi Zhan

Let M_n be the space of $n \times n$ complex matrices and let f be a function on M_n. A map $T : M_n \to M_n$ is said to *preserve* f if $f(T(A)) = f(A)$ for all $A \in M_n$.

The transpose operation ($T : A \mapsto A^t$ for all $A \in M_n$) and any given permutation similarity ($T : A \mapsto PAP^t$ for all $A \in M_n$ for a given permutation matrix P) preserve singular values and eigenvalues, and hence the rank and determinant; these maps all act on a matrix by rearranging some or all of its entries in a fixed pattern. Are there any other fixed permutations of the entries of a matrix that preserve these basic functions?